Reduction theory for connections over the formal punctured disc

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Abstract

We give a purely algebraic treatment of reduction theory for connections over the formal punctured disc. Our proofs apply to arbitrary connected linear algebraic groups over an algebraically closed field of characteristic 0. We also state and prove some new quantitative results.

Contents

1 Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Fix a connected linear algebraic group $\mathbf{G}$ over $\mathbf{k}$. Let $D^* := \operatorname{Spec} \mathbf{k}((t))$ denote the formal punctured disc over $\mathbf{k}$. In this paper we give an algebraic classification of formal $\mathbf{G}$-connections over D^* up to gauge equivalence. In order to achieve this we first prove that every connection can be put into canonical form [BV83] after passing to a ramified cover. Next, we describe a set of "good" representatives for canonical forms for which we can develop a clean description of Galois cohomology cocycles. We then proceed to describe said cocycles. As a consequence of our arguments we are also able to obtain some new quantitative results.

Our approach to the existence of canonical forms is based on the work of Babbitt and Varadarajan [BV83]. Some of the crucial parts in their argument are analytic in nature, so they only apply when the ground field is $\mathbb{C}$. We sidestep those parts to provide a completely algebraic proof. In addition, we simplify the global structure of their inductive arguments. We included detailed proofs of some the lemmas from [BV83] in order to keep the exposition self-contained.

Our treatment of uniqueness of canonical forms is substantially different from the one in [BV83]. We choose a different set of representatives for canonical classes in order to set up our Galois cohomology argument (see e.g. the list of properties in Theorem 5.3). This allows us to avoid the use of the complex exponential map. In our context the proof of uniqueness and the identification of the gauge transformation centralizer become algebraic arguments using power series.

We include separate treatments of reduction theory for connections in the reductive, the unipotent and the solvable case. We believe this is important, because the proofs are quite different depending on the type of group. This allows us to give sharper separate statements, including some new determinacy results in the unipotent and solvable cases (see Propositions 3.25, 3.29 and 5.2).

In Subsection 4.4 we give an explicit description of the reduction algorithm for reductive groups. As a byproduct of our arguments we obtain determinacy results for both the irregular and the regular part of the canonical form in the reductive case (Proposition 4.21). We also prove a new uniform bound on the ramification needed to put connections into canonical form (Proposition 4.19).

There is some related work by Schnürer. [Sch07] gives a purely algebraic proof of the reduction of formal connections when the group $\mathbf{G}$ is connected reductive and the connection has a regular singularity. He employs Galois cohomology methods to give concrete descriptions of gauge equivalences classes for the groups GL_n and SL_n. Our proofs of the existence of canonical forms apply to irregular connections for arbitrary connected linear algebraic groups. We give abstract Galois cohomology descriptions of gauge equivalence classes that also apply to any connected linear algebraic group (Subsection 5.3). We are able to give a concrete classification of the set of formal connections with regular singularities in Subsection 3.6.

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2 Some notation and definitions

2.1 Preliminaries on formal connections

We will always work over a fixed algebraically closed field $\mathbf{k}$ of characteristic 0. All schemes will be understood to be schemes over $\mathbf{k}$. An undecorated product of $\mathbf{k}$-schemes (e.g. $X \times S$) should always be interpreted as a fiber product over $\mathbf{k}$. $\mathbf{G}$ will be a connected linear algebraic group over $\mathbf{k}$ and $\mathfrak{g} = \text{Lie}(\mathbf{G})$ will be the corresponding Lie algebra. We let $\mathcal{O} = \mathbf{k}[[t]]$ denote the ring of formal power series over $\mathbf{k}$ and $F = \mathbf{k}((t))$ denote the corresponding field of Laurent series. $\mathcal{O}$ is a discrete valuation ring with maximal ideal $t\mathcal{O}$.

Recall that the module of Kähler differentials $\Omega^1_{\mathcal{O}/\mathbf{k}}$ classifies $\mathbf{k}$-derivations from $\mathcal{O}$. It is spanned as an $\mathcal{O}$-module by formal elements df for every $f \in \mathcal{O}$, subject to the relations $d(fg) = fdg + gdf$. We will work with the module of continuous Kähler differentials $\hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}$, which is defined as the completion

$$\hat{\Omega}^1_{\mathcal{O}/\mathbf{k}} := \varprojlim_{n} \Omega^1_{\mathcal{O}/\mathbf{k}} / \, t^n \, \Omega^1_{\mathcal{O}/\mathbf{k}}$$

This is a free $\mathcal{O}$-module of rank 1. The natural completion map $\widehat{(-)} : \Omega^1_{\mathcal{O}/\mathbf{k}} \to \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}$ can be thought of as the projection onto the quotient obtained by adding the extra relations coming from allowing termwise differentiation of power series.

Remark 2.1. *The module of ordinary Kähler differentials $\Omega^1_{\mathcal{O}/\mathbf{k}}$ is not finitely generated as an $\mathcal{O}$-module. We don't want to work with the sheaf $\Omega^1_{\mathcal{O}/\mathbf{k}}$, because the relations above do not include classical intuitive identities like $d(e^t) = e^t dt$. That is the reason why we use continuous Kähler differentials instead.*

For any positive natural number b, let $F_b := \mathbf{k}((t^{\frac{1}{b}}))$. This is a finite Galois extension of F with Galois group canonically isomorphic to μ_b, the group of b-roots of unity in $\mathbf{k}$. Under this isomorphism, we have that $\gamma \in \mu_b$ acts by $\gamma \cdot t^{\frac{1}{b}} = \gamma^{-1} t^{\frac{1}{b}}$. Notice that the choice of a primitive root of unity yields $\mu_b \cong \mathbb{Z}/b\,\mathbb{Z}$, since we are working in characteristic 0. A well known theorem of Puiseux states that the algebraic closure of F is $\overline{F} = \bigcup_{b \geq 1} F_b$.

In this paper we will work with a (right) $\mathbf{G}$-torsor P over the formal punctured disc $D^* := \operatorname{Spec} F$. We know that P can be trivialized, meaning that $P \cong \operatorname{Spec} F \times \mathbf{G}$ as right $\mathbf{G}$-torsors. This follows from theorems of Tsen and Springer, see [Ser02] page 80 - 3.3(b) and page 132 - 2.3(c). A formal connection A on P is a function from the set of trivializations of P into $\mathfrak{g} \otimes_{\mathbf{k}} \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$ that satisfies a certain transformation law. In order to describe the transformation law we need some notation.

We have a natural isomorphism $T_{\mathbf{G}} \cong \mathfrak{g} \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{G}}$ for the tangent sheaf given by left translation. Therefore, we get an isomorphism $\mathfrak{g} \otimes_{\mathbf{k}} \Omega^1_{\mathbf{G}/\mathbf{k}} \cong \mathfrak{g} \otimes_{\mathbf{k}} \operatorname{Hom}_{\mathcal{O}_{\mathbf{G}}}(T_{\mathbf{G}}, \mathcal{O}_{\mathbf{G}}) \cong \operatorname{Hom}_{\mathbf{k}}(\mathfrak{g}, \mathfrak{g}) \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{G}}$. The invariant $\mathfrak{g}$-valued 1-form on $\mathbf{G}$ that corresponds to $\mathrm{id}_{\mathfrak{g}} \otimes 1$ under this isomorphism is called the Maurer-Cartan form. We will denote it by $\omega \in \mathfrak{g} \otimes_{\mathbf{k}} \Omega^1_{\mathbf{G}/\mathbf{k}}$.

Suppose that we are given an element $g \in \mathbf{G}(F)$. We can think of it as a map $g : \operatorname{Spec} F \longrightarrow \mathbf{G}$. We can use g to pull back the Maurer-Cartan form to $\operatorname{Spec} F$ in order to obtain $g^*\omega \in \mathfrak{g} \otimes_{\mathbf{k}} \Omega^1_{F/\mathbf{k}} = \mathfrak{g} \otimes_{\mathbf{k}} \Omega^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$. By applying the completion map $\widehat{(-)} : \Omega^1_{\mathcal{O}/\mathbf{k}} \to \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}$, we get an element $\widehat{g^*\omega} \in \mathfrak{g} \otimes_{\mathbf{k}} \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$. Now we can define the gauge action of $\mathbf{G}(F)$ on $\mathfrak{g} \otimes_{\mathbf{k}} \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$. For any $g \in \mathbf{G}(F)$ and $B \in \mathfrak{g} \otimes_{\mathbf{k}} \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$, we set $g \cdot B := \operatorname{Ad}(g)B + \widehat{g^*\omega}$.

By a formal connection A for P we mean a function

$$A \; : \; \left\{ \text{trivializations } \; P \xrightarrow{\sim} \operatorname{Spec} F \times \mathbf{G} \right\} \;\longrightarrow\; \mathfrak{g} \otimes_{\mathbf{k}} \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$$

satisfying the following transformation law. Let $\phi_1, \phi_2 : P \xrightarrow{\sim} \operatorname{Spec} F \times \mathbf{G}$ be two trivializations of P. We know that $\phi_2 \circ \phi_1^{-1}$ is given by left multiplication by a unique element $g \in \mathbf{G}(F)$. We then require $A(\phi_2) = g \cdot A(\phi_1)$.

Remark 2.2. *Th reader might have encountered a different definition of formal connection. Using the action of* $\mathbf{G}$ *on* $\mathfrak{g} \otimes_{\mathbf{k}} \Omega^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$ *one can define a formal version of the Atiyah sequence [Ati57]. Splittings of such sequence will correspond to formal connections as we have defined them.*

Such a connection A is completely determined by its value at any given trivialization. We will often assume that we have chosen a fixed trivialization of P. Hence we can think of P as the trivial bundle, and think of A as the element of $\mathfrak{g} \otimes_{\mathbf{k}} \hat{\Omega}^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right]$ given by the image of this trivialization. Notice that we have implicitly fixed a choice of uniformizer t for $\mathcal{O}$. This yields an isomorphism $\hat{\Omega}^1_{\mathcal{O}/\mathbf{k}} = \mathcal{O}\, dt \cong \mathcal{O}$. We will often think of connections as elements of $\mathfrak{g}_F := \mathfrak{g} \otimes_{\mathbf{k}} F$ obtained under the induced isomorphism $\Omega^1_{\mathcal{O}/\mathbf{k}}\left[\frac{1}{t}\right] \cong F$.

All of the discussion above also applies over any finite field extension F_b of F. The choice of a uniformizer $u := t^{\frac{1}{b}}$ for F_b yields an isomorphism from F onto F_b sending t to u. This allows us to"lift" $\mathbf{G}$-bundles and trivializations from $\operatorname{Spec} F_b$ to $\operatorname{Spec} F$ by transport of structure. We can therefore lift connections from F_b to F.

There are some subtleties for the lift of connections when we think of them as elements of $\mathfrak{g}_F$. We generally take derivatives with respect to t, and not

$u = t^{\frac{1}{b}}$. That is, we fix the trancendental element $t = u^b$ of F_b in order to get the isomorphism $\hat{\Omega}^1_{\mathcal{O}_b/\mathbf{k}}\left[\frac{1}{u}\right] = (\mathcal{O}_b\, dt)\left[\frac{1}{u}\right] \cong \mathcal{O}_b\left[\frac{1}{u}\right] = F_b$. Under this identification, the lift of a $\mathbf{G}$-connection is not the obvious one given by replacing u by t. Instead, the lift of a connection $A = \sum_{j=r}^{\infty} A_j\, t^{\frac{j}{b}} \in \mathfrak{g}_{F_b}$ is given by $\tilde{A} := bt^{b-1}\sum_{j=r}^{\infty} A_j\, t^j$. This is called the b-lift of the connection.

Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus in $\mathbf{G}$. We will denote by $X_*(\mathbf{T})$ (resp. $X^*(\mathbf{T})$) the cocharacter (resp. character) lattice of $\mathbf{T}$. We will write $\langle -, - \rangle : X_*(\mathbf{T}) \otimes X^*(\mathbf{T}) \longrightarrow \mathbb{Z}$ for the canonical pairing. There is a natural inclusion $X_*(\mathbf{T}) \subset \mathrm{Lie}(\mathbf{T})$ given by taking differentials at the identity. We will freely use this identification without further notice. Note that a cocharacter $\lambda : \mathbb{G}_m \longrightarrow \mathbf{T} \subset \mathbf{G}$ yields a point $\lambda \in \mathbf{G}(\mathbf{k}[t, t^{-1}])$. We denote by t^{λ} the element of $\mathbf{G}(F)$ obtained via the natural inclusion $\mathbf{k}[t, t^{-1}] \hookrightarrow F$.

We will make use of the algebraic exponential map, as in [DG80] pg. 315. For $X \in t\mathfrak{gl}_n(\mathcal{O})$ we have an exponential $\exp(X) \in \mathbf{GL_n}(\mathcal{O})$ defined by $\exp(X) := \sum_{i=0}^{\infty} \frac{1}{i!} X^i$. By choosing a closed embedding $\mathbf{G} \hookrightarrow \mathbf{GL_n}$ we can similarly define an exponential map $\exp : t\mathfrak{g}(\mathcal{O}) \longrightarrow \mathbf{G}(\mathcal{O})$. It can be checked that this does not depend on the choice of embedding. We will only use one property of this map: for any $X \in \mathfrak{g}$, the image of $\exp(t^n\, X)$ when we reduce modulo t^{n+1} is given by $1 + t^n X \in \mathbf{G}\left(\mathcal{O}/t^{n+1}\mathcal{O}\right)$.

2.2 Adjoint orbits in semisimple Lie algebras

We include some facts about semisimple algebraic groups and their Lie algebras for future reference. Most of these results are standard and can be found in the book [CM93]. For the rest of this section we will assume that $\mathbf{G}$ is connected semisimple.

Recall that an element of a semisimple Lie algebra is called semisimple (resp. nilpotent) if the the image under the adjoint representation is semisimple (resp. nilpotent) as a linear transformation of $\mathfrak{g}$. It turns out that we can check these conditions on any faithful representation. This fact follows from the following theorem.

Theorem 2.3 (Additive Jordan Decomposition)**.** *Let $\mathfrak{g}$ semisimple. For any $A \in \mathfrak{g}$ there exist unique a A_s semisimple and A_n nilpotent such that*

(i) $A = A_s + A_n$

(ii) $[A_s, A_n] = 0$

Remark 2.4. *For a reductive Lie algebra, all elements of the center are considered semisimple. For the Lie algebra of an arbitrary linear algebraic group, we will usually fix a Levi subgroup $\mathbf{L}$ and speak of semisimple elements inside Lie($\mathbf{L}$).*

Recall that $\mathfrak{sl}_2 = \{X \in \mathfrak{gl}_2 \mid \mathrm{tr}(X) = 0\}$. The Lie bracket is given by the matrix commutator. We have $\mathfrak{sl}_2 = \mathbf{k}\langle H, X, Y\rangle$ as a vector space, where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. For the bracket we have the following identities:

$$
\begin{aligned}
[H,X] &= 2X \\
[H,Y] &= -2Y \\
[X,Y] &= H
\end{aligned}
$$

Definition 2.5. *An $\mathfrak{sl}_2$-**triple** in $\mathfrak{g}$ is a nonzero Lie algebra map $\phi : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$. We will often abuse notation and denote the images of H, X, Y with the same letters.*

Theorem 2.6 (Jacobson-Morozov)**.** *Let $\mathbf{G}$ be a connected semisimple algebraic group with Lie algebra $\mathfrak{g}$. Let $U \in \mathfrak{g}$ be a nilpotent element. Then there exists a homomorphism $\Phi : SL_2 \longrightarrow G$ such that the $\mathfrak{sl}_2$-triple corresponding to the differential $d\Phi : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ satisfies $d\Phi(Y) = U$. Moreover such a homomorphism is uniquely determined up to conjugation by an element of the centralizer $Z_{\mathbf{k}}(U)(\mathbf{k})$.*

If $Y \neq 0$ is a nilpotent element in $\mathfrak{g}$, we will denote by (H, X, Y) the $\mathfrak{sl}_2$-triple granted by Jacobson-Morozov. For any element $X \in \mathfrak{g}$, we will write $\mathfrak{g}_X$ for the centralizer of X in $\mathfrak{g}$.

We will let $G = \mathbf{G}(\mathbf{k})$ denote the $\mathbf{k}$-rational points of $\mathbf{G}$. Recall that for any $Y \in \mathfrak{g}$, the orbit under the adjoint action $\mathbf{G} \cdot Y$ can be equipped with the structure of a smooth locally closed subvariety of $\mathfrak{g}$. We will often harmlessly identify it with its closed points $G \cdot Y$. The following proposition is going to be the essential technical tool for the induction argument in the reductive case. The proof can be found in [BV83] pages 17-18.

Proposition 2.7. *Let $Y \neq 0$ be nilpotent in $\mathfrak{g}$. Let (H, X, Y) be the corresponding $\mathfrak{sl}_2$-triple. Then the affine space $Y + \mathfrak{g}_X$ meets the orbit $G \cdot Y$ exactly at Y. For any other nilpotent $U \in Y + \mathfrak{g}_X$ with $U \neq Y$, we have $dim(G \cdot U) > dim(G \cdot Y)$.*

Example 2.8. *If Y is regular nilpotent, then it is the unique nilpotent element in $Y + \mathfrak{g}_X$.*

Fix a maximal torus $\mathbf{T} \subset \mathbf{G}$. Let Φ be the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$. The coweight lattice $Q_{\mathbf{G}}$ of $\mathbf{G}$ with respect to $\mathbf{T}$ is defined to be $Q_{\mathbf{G}} := \mathrm{Hom}(\mathbb{Z}\Phi, \mathbb{Z})$. Since $\mathbf{G}$ is semisimple, the cocharacter lattice $X_*(\mathbf{T})$ has finite index in the coweight lattice $Q_{\mathbf{G}}$.

Definition 2.9. *The index $I(\mathbf{G})$ is defined to be the exponent of the finite group $Q_{\mathbf{G}} / X_*(\mathbf{T})$.*

Let $\Phi^\vee$ be the set of coroots of $\mathbf{G}$ with respect to $\mathbf{T}$. We have the following chain of inclusions

$$\mathbb{Z}\Phi^\vee \subset X_*(\mathbf{T}) \subset Q_{\mathbf{G}}$$

Definition 2.10. *We will denote by $J(\mathbf{G})$ the exponent of the finite group $Q_{\mathbf{G}} / \mathbb{Z}\Phi^\vee$.*

Remark 2.11. *Since all maximal tori in $\mathbf{G}$ are conjugate, $I(\mathbf{G})$ and $J(\mathbf{G})$ do not depend on the choice of $\mathbf{T}$.*

Let us fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ containing $\mathbf{T}$. This amounts a choice of positive roots Φ^+. We let Δ be the corresponding subset of simple roots.

Definition 2.12. *For a positive root $\alpha = \sum_{\beta \in \Delta} m_\beta \beta$, we define the height of α to be $hgt(\alpha) := \sum_{\beta\in\Delta} m_\beta$. Also, define the height of the Lie algebra $\mathfrak{g}$ to be $hgt(\mathfrak{g}) := sup_{\alpha \in \Phi^+} hgt(\alpha)$.*

To conclude this section, we define a function that measures the "size" of the semisimple element H in the Jacobson-Morozov triple corresponding to a nilpotent $Y \in \mathfrak{g}$. We can always arrange $H \in X_*(\mathbf{T})$. We will implicitly assume this from now on.

Definition 2.13. *Let $Y \in \mathfrak{g}$ be a nilpotent element. Let H be the corresponding semisimple element in the Jacobson-Morozov triple of Y. Then, we define $\Lambda(Y) := sup_{\alpha\in\Phi} \left(\frac{1}{2}\alpha(H) + 1\right)$. This function is constant on nilpotent orbits.*

Example 2.14. *Suppose that Y is regular nilpotent. We can choose H so that $\alpha(H) = 2$ for every $\alpha \in \Delta$ (see [CM93] Chapter 3). Therefore, $\Lambda(Y) = hgt(\mathfrak{g}) + 1$ in this case. It turns out that this is the biggest possible value for Λ. In other words $\Lambda(Y) \leq hgt(\mathfrak{g}) + 1$ for any nilpotent $Y \in \mathfrak{g}$.*

3 Regular connections

Fix a connected linear algebraic group $\mathbf{G}$ over $\mathbf{k}$. What we call regular connections are also known as connections with at worst regular singularities.

Definition 3.1. *A connection $A = \sum_{j=r}^\infty A_j\, t^j \in \mathfrak{g}_F$ is said to be of the first kind if if it has at most a simple pole at 0. This means that $r \geq -1$. A connection A is called regular if there exists $x \in \mathbf{G}(\overline{F})$ such that $x \cdot A$ is of the first kind.*

In the analytic context, regular connections are determined by topological data. Indeed, such connections are classified by their monodromy representation. Our goal in this section is to classify formal regular connections over an arbitrary ground field. We will start with the semisimple case.

3.1 Regular connections for semisimple groups

The main result of this subsection is the following.

Theorem 3.2. *Let $\mathbf{G}$ be a connected semisimple algebraic group. Let $A \in \mathfrak{g}_F$ be a regular connection. Then, there exists $x \in \mathbf{G}(\overline{F})$ such that $x \cdot A = t^{-1}C$ for some $C \in \mathfrak{g}$.*

We will say that a regular connection is in canonical form if it is of the form $t^{-1}C$ for some $C \in \mathfrak{g}$. In order to prove Theorem 3.2 we can assume that A is of first kind, because of the definition of regular connection. We first need the following definition and lemma, which actually work for arbitrary $\mathbf{G}$.

Definition 3.3. *Let $\mathbf{G}$ be a connected linear algebraic group. Let $A = \sum_{j=-1}^\infty A_j\, t^j$ be a connection of the first kind in $\mathfrak{g}_F$. The endomorphism $ad(A_{-1}) \in GL_n(\mathfrak{g})$ yields a decomposition of $\mathfrak{g}$ into generalized eigenspaces $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$. We say that A is aligned if $A_j \in \mathfrak{g}_{j+1}$ for all j.*

Lemma 3.4. *Let $\mathbf{G}$ be a connected linear algebraic group and $A = \sum_{j=-1}^{\infty} A_j\, t^j$ a formal connection of the first kind in $\mathfrak{g}_F$. Then there exist $x \in \mathbf{G}(\mathcal{O})$ such that $x \cdot A$ is aligned.*

Proof. We will build inductively a sequence $(B_j)_{j=1}^{\infty}$ of elements of $\mathfrak{g}$ such that the change of trivialization by $x := \lim_{n\to\infty} \prod_{j=0}^{n-1} \exp(t^{n-j}\, B_{n-j})$ puts A in aligned form.

Let $k \in \mathbb{N}$. Suppose that we have chosen $B_j \in \mathfrak{g}$ for all $j \leq k$ such that the connection $A^{(k)} = \sum_{l=-1}^{\infty} A_l^{(k)}\, t^l$ defined by $A^{(k)} := \prod_{j=0}^{k-1} \exp(t^{k-j}\, B_{k-j}) \cdot A$ satisfies $A_l^{(k)} \in \mathfrak{g}_{l+1}$ for all $l < k$. The base case $k = 0$ is trivial. Note that we will have $A_{-1}^{(k)} = A_{-1}$. Let's try to determine B_{k+1}.

Recall that $\exp(t^{k+1}\, B_{k+1}) \equiv 1 + t^{k+1}\, B_{k+1} \pmod{t^{k+2}}$. By an elementary matrix computation (choose an embedding of $\mathbf{G} \hookrightarrow \mathrm{GL}_{\mathbf{n}}$), we can see that

$$\exp(t^{k+1}B_{k+1}) \cdot A^{(k)} \equiv \sum_{l=-1}^{k-1} A_l^{(k)}\, t^l + [A_k^{(k)} - (ad(A_{-1}) - (k+1))B_{k+1}]\, t^k \pmod{t^{k+1}}$$

Decompose $\mathfrak{g}$ into generalized $ad(A_{-1})$ eigenspaces $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}$. By definition the operator $ad(A_{-1}) - (k+1)$ restricts to an automorphism of $\mathfrak{g}_{\lambda}$ for all $\lambda \neq k+1$. In particular, we can choose $B_{k+1} \in \mathfrak{g}$ such that $A_k^{(k)} - (ad(A_{-1}) - (k+1))B_{k+1}$ is in $\mathfrak{g}_{k+1}$. By induction, we are done with the construction of the sequence $(B_j)_{j=1}^{\infty}$. It is now follows by construction that the gauge transformation by $x := \lim_{n\to\infty} \prod_{j=0}^{n-1} \exp(t^{n-j}\, B_{n-j})$ puts A in aligned form. □

Remark 3.5. *The aligned connection is actually in $\mathfrak{g} \otimes \mathbf{k}[t, t^{-1}]$. The coefficient with largest exponent is $(x \cdot A)_j\, t^j$, where $j + 1$ is the biggest integer eigenvalue of $ad\,(\,(A_{-1})_s\,)$. We denote this number by $k(A_{-1}) := j + 1$ for further reference. In order to determine the resulting aligned connection, we only need to multiply by $k(A_{-1})$-many exponentials in the proof above. Therefore the aligned form only depends on A_j for $-1 \leq j \leq k(A_{-1})$. Notice that $k(A_{-1})$ can drastically change if we scale A by a scalar in $\mathbf{k}$. This is a reflection of the fact that gauge transformations are not $\mathbf{k}$-linear.*

Example 3.6. *Suppose that $ad\,(\,(A_{-1})_s\,)$ does not have any integer eigenvalues. Then the aligned connection will be in canonical form.*

Now we can finish the proof of the theorem.

Proof of Theorem 3.2. By Lemma 3.4, we can assume that A is an aligned connection in $\mathfrak{g}_F$. Let $(A_{-1})_s$ be the semisimple part of A_{-1}. Choose a maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that the corresponding Cartan subalgebra Lie($\mathbf{T}$) contains $(A_{-1})_s$. Fix a choice of positive roots Φ^+ of $\mathbf{G}$ relative to $\mathbf{T}$. Let Δ be the subset of simple roots. Choose a basis for $\mathbf{k}$ as a vector space over $\mathbb{Q}$. Suppose that 1 is one of the basis elements. Let $\pi : \mathbf{k} \longrightarrow \mathbb{Q}$ be the corresponding projection. We can define τ in Lie($\mathbf{T}$) given by $\tau(\alpha) = \pi(\alpha((A_{-1})_s))$ for all $\alpha \in \Delta$.

There exists $b \in \mathbb{N}$ such that $b\tau$ is in the cocharacter lattice of $\mathbf{T}$. We let $\mu := b\tau$ be the corresponding cocharacter. Recall from the preliminaries that we have a b-lift $\tilde{A} = \sum_{j=-1}^{\infty} bA_j\, t^{bj+b-1}$. We can assume that we are working with $\tilde{A}$ by passing to the b-ramified cover. We claim that $t^{-\mu} \cdot \tilde{A}$ is in canonical form. In order to show this, it is convenient to use the *ad* representation and view everything as matrices in $\mathrm{End}(\mathfrak{g})$. The root decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ gives us the spectral decomposition of $(A_{-1})_s$.

We can view $\mathrm{Ad}(t^{-\mu})$ as a matrix in $\mathrm{GL}(\mathfrak{g}_F)$. $\mathrm{Ad}(t^{-\mu})$ acts as the scalar $t^{\langle -\mu, \beta\rangle}$ on the root space $\mathfrak{g}_\beta$. By assumption A is aligned. This means that A_j is in a sum of root spaces $\mathfrak{g}_\beta$ where $(A_{-1})_{j+1}$ has eigenvalue $j+1$. These are the root spaces $\mathfrak{g}_\beta$ where $\beta((A_{-1})_s) = j+1$. By the construction of μ, we know that $\langle \mu, \beta\rangle = b\beta((A_{-1})_s)$ whenever $\beta((A_{-1})_s)$ is an integer. Therefore, $\mathrm{Ad}(t^{-\mu})\, A_j = t^{-bj-b} A_j$. We conclude that

$$t^{-\mu} \cdot \tilde{A} = t^{-\mu} \cdot \left(\sum_{j=-1}^{\infty} bA_j\, t^{bj+b-1} \right) = \left(\sum_{j=-1}^{\infty} bA_j \right) t^{-1} + \frac{d}{dt} \left(t^{-\mu}\right) t^{\mu}$$

For the last term $\frac{d}{dt}\left(t^{-\mu}\right) t^{\mu}$ we are performing the calculation in $\mathrm{End}(\mathfrak{g}_F)$. A matrix computation yields $\frac{d}{dt}\left(t^{-\mu}\right) t^{\mu} = -\mu\, t^{-1}$. The theorem follows. ☐

Remark 3.7. *Notice that we needed to pass to a ramified cover because the functional τ we defined didn't necessarily live in the cocharacter lattice. Babbitt and Varadarajan prove the theorem using the analytic theory of regular singular connections (see section 8 of [BV83]). In the analytic setting, we need to pass to a ramified cover whenever the monodromy class of the connection is not in the image of the exponential map. This is sometimes a sharper condition in the reductive case. For example all conjugacy classes in $\mathbf{GL}_n$ are exponential, so we don't need to pass to a ramified cover to reduce regular $\mathbf{GL_n}$-connections. This can also be proven algebraically using the center of $\mathbf{GL}_n$. See the argument in pages 19-22 of [BV83].*

Remark 3.8. *We only need to fix a rational basis of span$_{\mathbb{Q}}\{\alpha((A_{-1})_s) : \alpha \in \Delta\}$ in the proof above. So the argument is constructive.*

We can be a bit more careful in the proof of Theorem 3.2. This way we can get a uniform bound for the ramification needed. We record this as a small lemma for future reference.

Lemma 3.9. *We can always choose $b \leq hgt(\mathfrak{g}) \cdot I(\mathbf{G})$ in the proof of Theorem 3.2.*

Proof. We can set $\tau(\alpha)$ to be the best approximation of $\pi(\alpha((A_{-1})_s))$ in $\frac{1}{\mathrm{hgt}(\mathfrak{g})}\mathbb{Z}$. By the definition of $\mathrm{hgt}(\mathfrak{g})$, it follows that $\tau(\beta) = \beta((A_{-1})_s)$ whenever $\beta((A_{-1})_s)$ is an integer. So the proof of Theorem 3.2 still goes through with this choice of τ.

By construction we have $\mathrm{hgt}(\mathfrak{g})\tau \in Q_{\mathbf{G}}$. Then the definition of $I(\mathbf{G})$ implies thtat $\mathrm{hgt}(\mathfrak{g}) I(\mathbf{G})\tau \in X_*(\mathbf{T})$. Hence we can choose $b = \mathrm{hgt}(\mathfrak{g}) I(\mathbf{G})$. ☐

Choose a maximal torus $\mathbf{T} \subset \mathbf{G}$. Let W be the Weyl group of $\mathbf{G}$ with respect $\mathbf{T}$. Fix a projection $\pi : \mathbf{k} \longrightarrow \mathbb{Q}$ as in the proof above. We can extend this projection to a natural map $\pi : \mathrm{Lie}(\mathbf{T}) \cong X_*(\mathbf{T}) \otimes \mathbf{k} \longrightarrow X_*(\mathbf{T}) \otimes \mathbb{Q}$. We will once and for all fix a fundamental domain $\mathfrak{D}$ for the action of W on the set

$\Xi := \{C \in \mathrm{Lie}(\mathbf{T}) \mid \pi(C) = 0\}$. Notice that Ξ is a set of representatives for the quotient $\mathrm{Lie}(\mathbf{T}) / X_*(\mathbf{G}) \otimes \mathbb{Q}$.

We can always choose x in Theorem 3.2 so that the semisimple part C_s is in $\mathfrak{D}$. We record this as a corollary.

Corollary 3.10. *Let $\mathbf{G}$ be connected semisimple with a choice of maximal torus $\mathbf{T} \subset \mathbf{G}$. Let $A \in \mathfrak{g}_F$ be a regular connection. Then, there exists $x \in \mathbf{G}(\overline{F})$ such that $x \cdot A = t^{-1}C$ for some $C \in \mathfrak{g}$ satisfying $C_s \in \mathfrak{D}$.*

Proof. By Theorem 3.2, we can assume that $A = t^{-1}\, C$ for some $C \in \mathfrak{g}$. Since $\mathbf{k}$ is algebraically closed, we can conjugate the semisimple element C_s to the torus $\mathbf{T}$. By applying the gauge transformation $t^{-\pi(C_s)}$, we can assume that $\pi(C_s) = 0$. Finally, we can conjugate by an element of W to obtain $C_s \in \mathfrak{D}$. □

The following proposition will be crucial in establishing uniqueness of canonical reductions in general.

Proposition 3.11. *Let $\mathbf{G}$ be connected semisimple with a choice of maximal torus $\mathbf{T} \subset \mathbf{G}$. Let $C, D \in \mathfrak{g}$ with $C_s, D_s \in \mathfrak{D}$. Suppose that there exists $x \in \mathbf{G}(\overline{F})$ such that $x \cdot (t^{-1}\, C) = t^{-1}\, D$. Then we have $C_s = D_s$. Moreover x is a $\mathbf{k}$-point in the centralizer $Z_{\mathbf{G}}(C_s)(\mathbf{k})$.*

Proof. By lifting everything to a ramified cover, we can assume for simplicity that $x \in \mathbf{G}(F)$. Choose a faithful representation $\mathbf{G} \hookrightarrow \mathrm{GL}_{\mathbf{n}}$. We can view $x \in \mathrm{GL}_{\mathbf{n}}(F)$ and $C, D \in \mathfrak{gl}_n$.

Let's consider the linear transformation U in $\mathrm{End}(\mathfrak{gl}_n)$ given by $U\, v = Dv - vC$ for all $v \in \mathfrak{gl}_n$. Notice that we can write $U = U_s + U_n$, where

$$U_s\, v := D_s v - v C_s$$
$$U_n\, v := D_n v - v C_n$$

We know that C_s and D_s can be simultaneously diagonalized. Therefore U_s is semisimple. The eigenvalues of U_s are differences of eigenvalues of C_s and D_s. Since $\pi(C_s) = \pi(D_s) = 0$, we conclude that 0 is the only possible rational eigenvalue of U_s. By definition, we have that U_n is nilpotent and $[U_s, U_n] = 0$. We conclude that $U = U_s + U_n$ is the additive Jordan decomposition of U. In particular the set of eigenvalues of U is the same as the set of eigenvalues of U_s. Therefore, 0 is the only possible rational eigenvalue of U.

The condition $x \cdot (t^{-1}\, C) = t^{-1}\, D$ can be expressed as $\frac{d}{dt} x = t^{-1} U\, x$. Here we are viewing x as an invertible matrix in $\mathfrak{gl}_n(F)$. Set $x = \sum_{j=r}^{\infty} x_j\, t^j$. Then this condition reads

$$\sum_{j=r}^{\infty} j x_j\, t^{j-1} = \sum_{j=r}^{\infty} U\, x_j\, t^{j-1}$$

Hence we have $j x_j = U\, x_j$ for all j. Since 0 is the only possible rational eigenvalue of U, we conclude that $x_j = 0$ for all $j \neq 0$. Therefore, $x = x_0 \in \mathbf{G}(\mathbf{k})$. Hence the relation $x \cdot (t^{-1}\, C) = t^{-1}\, D$ implies that $\mathrm{Ad}(x)\, C = D$. By uniqueness of Jordan decomposition for $\mathrm{GL}_{\mathbf{n}}$, this means that $\mathrm{Ad}(x)\, C_s = D_s$.

$\mathrm{Lie}(\mathbf{T})/W$ parametrizes semisimple conjugacy classes in $\mathfrak{g}$ (see [CM93] Chapter 2). In particular, $\mathfrak{D}$ is a set of representatives of conjugacy classes of semisimple elements that map to 0 under π. We conclude that we must have $C_s = D_s$. Now $\mathrm{Ad}(x)\, C_s = D_s$ implies that $x \in Z_{\mathbf{G}}(C_s)(\mathbf{k})$. $\square$

3.2 Regular connections for tori and reductive groups

Let us prove the analogous theorem for tori.

Proposition 3.12. *Let $\mathbf{G}$ be a torus and $A = \sum_{j=-1}^{\infty} A_j\, t^j$ a formal connection of the first kind. Then there exists $x \in \mathbf{G}(\mathcal{O})$ such that $x \cdot A = t^{-1} A_{-1}$. Moreover, there is a unique such x with $x \equiv 1 \pmod t$.*

Proof. Since $\mathbf{k}$ is algebraically closed, $\mathbf{G}$ is split. Therefore the theorem follows from the special case $\mathbf{G} = \mathbb{G}_m$. We are reduced to an elementary computation. Let $v = \sum_{j=0}^{\infty} A_j t^j \in \mathcal{O}$. It suffices to find $u = \sum_{j=0}^{\infty} B_j t^j \in \mathcal{O}^\times$ with $\frac{d}{dt}(u) = -vu$. By expanding we see that we want $(j+1)B_{j+1} = -\sum_{l=0}^{j} A_l B_{j-l}$ for all $j \geq 0$. This is a recurrence we can solve, because we are in characteristic 0. We can set the initial condition $B_0 = 1$ and then the rest of the coefficients are uniquely determined. $\square$

Example 3.13. *When $\mathbf{G} = \mathbb{G}_m$ we can phrase this result concretely in terms of differential equations. In this case we have an equation $\frac{d}{dt}x = Ax$, where $A \in \mathbf{k}(\!(t)\!)$ is a Laurent series with at worst a simple pole. The statement says that we can do a multiplicative change of variables $y = Bx$ for some power series $B \in \mathcal{O}^\times$ such that the equation becomes $\frac{d}{dt}y = \frac{a}{t}y$ for some scalar $a \in \mathbf{k}$. So all solutions to regular singular equations of degree one look like $f(t)t^a$ for some $f(t) \in \mathcal{O}^\times$ and $a \in \mathbf{k}$.*

Let us state a uniqueness result for canonical forms of regular formal connections on tori.

Proposition 3.14. *Let $\mathbf{G}$ be a torus, and let $C_1, C_2 \in \mathfrak{g}$. Suppose that there exists $x \in \mathbf{G}(F)$ with $x \cdot (t^{-1}\, C_1) = t^{-1}\, C_2$. Then, we have $x = g\, t^{\mu}$ for some cocharacter $\mu \in X_*(\mathbf{G})$ and some $g \in \mathbf{G}(\mathbf{k})$. In this case $C_1 = C_2 - \mu$.*

Proof. We will do the computation for $\mathbf{G} = \mathbb{G}_m$. The general case follows from the same argument. Write $x = k\, t^r\, y$, where $k \in \mathbf{k}^\times$ and $y = 1 + \sum_{j=1}^{\infty} a_j\, t^j$. Then,

$$x \cdot \left(t^{-1}\, C_1\right) \;=\; t^{-1}\, C_1 + rt^{-1} + dy\, y^{-1} \;=\; t^{-1}\, C_2$$

Notice that $dy\, y^{-1}$ is in $\mathbf{k}[\![t]\!]$. By looking at the nonnegative coefficients in the equation above, we conclude that $dy = 0$. Therefore we have $y = 1$. Hence $x = k\, t^r$, and the result follows. $\square$

We can patch together some previous of results to get canonical forms for regular connections when the group is reductive.

Corollary 3.15. *Let* $\mathbf{G}$ *be reductive and* $A \in \mathfrak{g}_F$ *a regular formal connection. Then there exists* $x \in \mathbf{G}(\overline{F})$ *such that* $x \cdot A = t^{-1}C$ *for some* $C \in \mathfrak{g}$.

Proof. We can assume that A is of the first kind. Let $\mathbf{Z}_G^0$ be the neutral component of the center of $\mathbf{G}$. Set $\mathfrak{z} := \mathrm{Lie}(\mathbf{Z}_{\mathbf{G}}^0)$. Let $\mathbf{G}_{\mathrm{der}}$ the derived subgroup of $\mathbf{G}$. $\mathbf{G}_{der}$ is semisimple with Lie algebra $\mathfrak{g}_{\mathrm{der}} := [\mathfrak{g}, \mathfrak{g}]$. We have $\mathfrak{g} = \mathfrak{g}_{\mathrm{der}} \oplus \mathfrak{z}$. Decompose $A = A_{\mathfrak{g}_{\mathrm{der}}} + A_{\mathfrak{z}}$. By the semisimple case there exists $x \in \mathbf{G}_{\mathrm{der}}(\overline{F})$ such that $x \cdot A_{\mathfrak{g}_{\mathrm{der}}}$ is in canonical form. Now $x \cdot A = x \cdot A_{\mathfrak{g}_{\mathrm{der}}} + A_{\mathfrak{z}}$. Use the result for tori to put $A_{\mathfrak{z}}$ in canonical form and conclude. □

Remark 3.16. *By Remark 3.5, we only need to know* $k\left((A_{\mathfrak{g}_{der}})_{-1} \right)$*-many coefficients of a connection of the first kind in order to determine its canonical form. The bound for the ramification needed in this case is reduced to the bound for the semisimple group* $\mathbf{G}_{der}$ *as explained in Lemma 3.9.*

Let us mention some uniqueness statements for regular connections in reductive groups. Notice that the setup before Corollary 3.10 applies to the reductive case. We formulate the analogous statement for convenience.

Corollary 3.17. *Let* $\mathbf{G}$ *connected reductive with maximal torus* $\mathbf{T} \subset \mathbf{G}$.

(i) Let $A \in \mathfrak{g}_F$ *be a regular connection. Then there exists* $x \in \mathbf{G}(\overline{F})$ *such that* $x \cdot A = t^{-1}C$ *for some* $C \in \mathfrak{g}$ *satisfying* $C_s \in \mathfrak{D}$.

(ii) Assume that $C, D \in \mathfrak{g}$ *satisfy* $C_s, D_s \in \mathfrak{D}$. *Suppose that there exists* $x \in \mathbf{G}(\overline{F})$ *such that* $x \cdot (t^{-1}\, C) = t^{-1}\, D$. *Then, we have* $C_s = D_s$. *Moreover* x *is in the centralizer* $Z_{\mathbf{G}}(C_s)(\mathbf{k})$.

Proof. Part (i) follows by combining Proposition 3.12 for tori and Corollary 3.10 for semisimple groups. Part (ii) follows from the same argument as in Proposition 3.11. □

This corollary allows us to give a concrete parametrization of regular $\mathbf{G}(\overline{F})$-gauge equivalence classes of formal connections. Let $A \in \mathfrak{g}_F$ be a regular formal connection. Suppose that $B = t^{-1}C$ is a connection in canonical form that is $\mathbf{G}(\overline{F})$-gauge equivalent to A. Assume that $C_s \in \mathfrak{D}$. By Corollary 3.17, C_s does not depend on the choice of canonical form B. Let W denote the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$. Recall that $\mathfrak{D}$ is a set of representatives for $(X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$. In particular we get a well defined element in $(X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$ corresponding to C_s.

Definition 3.18. *Let* $A \in \mathfrak{g}_F$ *be a regular formal connection as above. We define the semisimple* $\overline{F}$*-monodromy of* A *to be the element* $m^s_{A,\overline{F}} \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$ *corresponding to* C_s *as described above.*

Let $m \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$. We define $Z_{\mathbf{G}}(m)$ to be the centralizer in $\mathbf{G}$ of the unique representative of m in $\mathfrak{D}$. It turns out that $Z_{\mathbf{G}}(m)$ is a Levi subgroup of a parabolic in $\mathbf{G}$. It is well known that the Lie algebra centralizer of a semisimple element $\mathrm{Lie}(Z_{\mathbf{G}} = \mathfrak{g}^m$ is the Levi component of a parabolic subalgebra of $\mathfrak{g}$. For connectedness, we can pass to an isogenous cover $p : \tilde{\mathbf{G}} \longrightarrow \mathbf{G}$ with simply connected derived subgroup. Notice that $p(Z_{\tilde{\mathbf{G}}}(m)) = Z_{\mathbf{G}}(m)$. So it suffices to prove connectedness of $Z_{\tilde{\mathbf{G}}}(m)$, which follows from [Hum95] pg. 33. Note that the isomorphism class of $Z_{\mathbf{G}}(m)$ does not depend on the choice of projection $\pi : \mathbf{k} \longrightarrow \mathbb{Q}$ and fundamental domain $\mathfrak{D}$. In fact, $Z_{\mathbf{G}}(m) \cong Z_{\mathbf{G}}(C)$ for any representative C such that $\mathrm{ad}(C)$ has no rational eigenvalues.

Corollary 3.17 implies that the nilpotent part of a canonical form B that is $\mathbf{G}(\overline{F})$-gauge equivalent to A is uniquely determined up to $Z_{\mathbf{G}}(m^s_{A,\overline{F}})$-conjugacy. We record this as a corollary.

Corollary 3.19. *Fix $m \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$. Let $\mathcal{N}_{Z_{\mathbf{G}}(m)}$ denote the nilpotent cone in the Lie algebra of $Z_{\mathbf{G}}(m)$. There is a natural correspondence*

$$\left\{\text{regular } A \in \mathfrak{g}_{\overline{F}} \text{ with } m^s_{A,\overline{F}} = m\right\} / \,\mathbf{G}(\overline{F}) \quad \longleftrightarrow \quad \mathcal{N}_{Z_{\mathbf{G}}(m)} /\, Z_{\mathbf{G}}(m)$$

Let $S \subset \Delta$ be a subset of simple roots. Each $\alpha \in S$ induces a linear functional $X_*(\mathbf{T}) \otimes \mathbf{k} \longrightarrow \mathbf{k}$. Let H_α be the hyperplane of $X_*(\mathbf{T}) \otimes \mathbf{k}$ where this functional vanishes. We denote by $\overline{H}_\alpha$ the image of H_α in $(X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$. Define $\overline{H}_S := \bigcap_{\alpha \in S} \overline{H}_\alpha$. We say that $m \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$ is of type S if we have $m \in \overline{H}_S$ and $m \notin \overline{H}_V$ for all $S \subsetneq V \subset \Delta$. Let Q_S be the set of all m of type S. For any $m \in Q_S$, the centralizer $Z_{\mathbf{G}}(m)$ is conjugate to the standard Levi $\mathbf{M}_S$ associated to the subset of simple roots $S \in \Delta$. We get the following rewording of the corollary above.

Corollary 3.20. *There is a natural correspondence*

$$\{\text{regular formal connections}\}\, / \,\mathbf{G}(\overline{F}) \quad \longleftrightarrow \quad \bigsqcup_{S \subset \Delta} Q_S \times \mathcal{N}_{\mathbf{M}_S} /\, \mathbf{M}_S$$

This gives us a procedure to describe all regular $\mathbf{G}(\overline{F})$-gauge equivalence classes of formal connections. For each $S \subset \Delta$, the group $\mathbf{M}_S$ is connected and reductive. It turns out that the set of nilpotent orbits $\mathcal{N}_{\mathbf{M}_S} /\, \mathbf{M}_S$ is a finite set. It admits many well studied parametrizations. For example, nilpotent orbits can be classified by Dynkin-Kostant diagrams as in [CM93] Chapter 3.

3.3 Connections for unipotent groups

All connections in a unipotent group are regular. They can also be put into canonical form.

Proposition 3.21. *Let $\mathbf{G}$ be connected unipotent and let $A \in \mathfrak{g}_F$ be a connection. Then, there exists $x \in \mathbf{G}(F)$ such that $x \cdot A = t^{-1}C$ for some $C \in \mathfrak{g}$.*

Proof. We proceed by induction on $\dim(\mathbf{G})$. Suppose that $\dim(\mathbf{G}) = 1$. Since $\mathrm{char}(\mathbf{k}) = 0$, we know that $\mathbf{G} \cong \mathbb{G}_a$. In this case the theorem follows from an elementary computation similar to the one in the proof of Proposition 3.12. See Example 3.23 below.

Now assume $\dim(\mathbf{G}) \geq 2$. Since $\mathbf{k}$ is of characteristic 0, $\mathbf{G}$ is split unipotent. In particular, the center $\mathbf{Z_G}$ contains a closed subgroup $\mathbf{H}$ isomorphic to $\mathbb{G}_a$. Let $\mathrm{Lie}(\mathbf{H}) = \mathfrak{h}$. By induction there exists $\overline{x} \in \mathbf{G}/\mathbf{H}(F)$ such that $\overline{x} \cdot \overline{A} \in \mathfrak{g}/\mathfrak{h}$ is in canonical form. We can lift $\overline{x}$ to an element $x \in \mathbf{G}(F)$ because $H^1\big(F, \mathbf{H}(\overline{F})\big) = H^1\big(F, \overline{F}\big) = 0$ (this is sometimes called the additive version of Hilbert's Theorem 90). By construction, we have $x \cdot A = t^{-1}C + B$ for some $C \in \mathfrak{g}$ and $B \in \mathfrak{h}_F$. Now we can use the base case for $\mathbf{H} \cong \mathbb{G}_a$ to put B into regular canonical form. $\square$

Remark 3.22. *Here we didn't need to pass to a ramified cover in order to find a good trivialization.*

Example 3.23. *In the case of $\mathbf{G} = \mathbb{G}_a$, we can phrase this result concretely in terms of differential equations. We use the embedding $\mathbb{G}_a \hookrightarrow GL_2$, so that we can interpret the connection as system of differential equations*

$$\frac{d}{dt}x_1 = Ax_2$$
$$\frac{d}{dt}x_2 = 0$$

We can ignore the second equation. Set $x_2 = c$, where $c \in \mathbf{k}$ is a constant. We are left with the inhomogeneous equation $\frac{d}{dt}x_1 = cA$ for some Laurent series cA. In this case the statement reduces to the obvious fact that we can find a formal antiderivative for any Laurent series with residue 0 *(i.e. $A_{-1} = 0$).*

We now prove uniqueness of the canonical form up to conjugacy.

Proposition 3.24. *Let $\mathbf{U}$ be a unipotent group. Let C_1, C_2 be two elements of the Lie algebra $\mathfrak{u} := Lie(\mathbf{U})$. Suppose that there exists $x \in \mathbf{U}(F)$ such that $x \cdot (t^{-1}C_1) = t^{-1}C_2$. Then, we have $x \in \mathbf{U}(\mathbf{k})$.*

Proof. We will argue by induction on the dimension of $\mathbf{U}$. If $\dim(\mathbf{U}) = 1$, then $\mathbf{U} \cong \mathbb{G}_a$. We can write $x = \sum_{j=m}^{\infty} a_j \, t^j$ for some $a_j \in \mathbf{k}$. The hypothesis then becomes

$$x \cdot \big(t^{-1}C_1\big) = t^{-1}C_1 + dx = t^{-1}C_2$$

This means that $dx = \sum_{j=m}^{\infty} j a_j \, t^{j-1} = t^{-1}\,(C_2 - C_1)$. In particular we must have $ja_j = 0$ for all $j \neq 0$. Hence $x = a_0 \in \mathbf{k}$.

Suppose that $\mathbf{U}$ is an arbitrary unipotent group. Assume that the result holds for all unipotent groups of smaller dimension. Let $\mathbf{H}$ be a subgroup of the center $Z_{\mathbf{U}}$ of $\mathbf{U}$ such that $\mathbf{H} \cong \mathbb{G}_a$ (this is possible because $\mathrm{char}(\mathbf{k}) = 0$) . Let $\overline{x} \in \mathbf{U}/\mathbf{H}\,(F)$ be the image of x in the quotient. By the induction hypothesis, the proposition holds for $\mathbf{U}/\,\mathbf{H}$. Hence we have that $\overline{x} \in \mathbf{U}/\mathbf{H}\,(\mathbf{k})$. We can lift $\overline{x}$ to an element $v \in \mathbf{U}(\mathbf{k})$, since $\mathbf{k}$ is algebraically closed.

We can therefore write $x = vu$, with $u \in \mathbf{H}(F)$. Our assumption thus becomes

$$x \cdot \big(t^{-1}C_1\big) \;=\; t^{-1}\mathrm{Ad}(v)\,\mathrm{Ad}(u)\,C_1 + \mathrm{Ad}(v)du \;=\; t^{-1}C_2$$

Since $u \in \mathbf{H}(F) \subset Z_{\mathbf{U}}(F)$, we have $\mathrm{Ad}(u)C_1 = C_1$. After rearranging we get

$$du \;=\; t^{-1}\left(\mathrm{Ad}(v^{-1})\,C_2 - C_1\right)$$

The computation for $\mathbb{G}_a$ above implies that $u \in \mathbf{H}(\mathbf{k})$. Therefore $x \in \mathbf{U}(\mathbf{k})$. $\square$

We end this section with a determinacy result for canonical forms in the unipotent case. Recall that the nilpotency class of a unipotent group is the length of the upper central series. For example, a commutative unipotent group has nilpotency class 0.

Proposition 3.25. *Let $\mathbf{U}$ be a unipotent group of nilpotency class n. Let $A = \sum_{j=m}^{\infty} A_j\, t^j \in \mathfrak{u}_F$ be a connection with $A_m \neq 0$.*

(i) If $m > -1$, then there exists $x \in \mathbf{U}(\mathcal{O})$ such that $x \equiv 1 \pmod{t^{m+1}}$ and $x \cdot A = 0$.

(ii) If $m \leq -1$, then the gauge equivalence class of A is determined by A_j for $m \leq j < n(|m|-1)$. More precisely, suppose that B is another connection with $B \equiv A \, \left(mod\, t^k\right)$ for some $k \geq n(|m|-1)$. Then there exists $x \in \mathbf{U}(\mathcal{O})$ with $x \equiv 1 \left(mod\, t^{k-n|m|+n+1}\right)$ such that $x \cdot A = B$.

Proof. We will induct on the nilpotency class n. The base case $n = 0$ means that $\mathbf{U} \cong \mathbb{G}_a^l$ for some l. Here we can make use of the explicit computation we have done for $\mathbb{G}_a$ a few times already. Define $u_A := -\sum_{j=0}^{\infty} \frac{1}{j+1} A_{j+1}\, t^{j+1}$. We have

$$u_A \cdot A = A + du_A = \sum_{j=m}^{-1} A_j\, t^j$$

Now both (i) and (ii) are clear by taking $x = -u_B + u_A$ (we use $B = 0$ for part (i)).

For the induction step, let $\mathbf{U}$ be an arbitrary unipotent group of nilpotency class n. We will think of $\mathbf{U}$ as embedded in the group of upper triangular matrices of GL_p for some p. By definition, the quotient $\mathbf{U}/\, Z_{\mathbf{U}}$ of $\mathbf{U}$ by its center $Z_{\mathbf{U}}$ has nilpotency class $n-1$. It follows from the matrix description that we can choose a section s over $\mathbf{k}$ for the $Z_{\mathbf{U}}$-torsor $\mathbf{U} \longrightarrow \mathbf{U}/\, Z_{\mathbf{U}}$ such that $s(1) = 1$ (this is a section as $\mathbf{k}$-schemes, it is not a homomorphism).

Let's address part (i). Let $\overline{A}$ be the image of A in the quotient $\mathrm{Lie}(\mathbf{U}/\, Z_{\mathbf{U}})_F$. By the induction hypothesis, there exists $\overline{x} \in \mathbf{U}/\, Z_{\mathbf{U}}(\mathcal{O})$ such that $\overline{x} \equiv 1 \pmod{t^{m+1}}$ and $\overline{x} \cdot \overline{A} = 0$. Therefore, we have $s(\overline{x}) \cdot A \in \mathrm{Lie}(Z_{\mathbf{U}})_F$. Notice that we have $s(\overline{x}) \equiv s(\overline{x})^{-1} \equiv 1 \pmod{t^{m+1}}$. It follows that

$$s(\overline{x}) \cdot A \;=\; s(\overline{x})\, A\, s(\overline{x})^{-1} \;+\; d\, s(\overline{x})\; s(\overline{x})^{-1} \;\equiv 0 \pmod{t^m}$$

Now we can conclude by using the base case for $Z_{\mathbf{U}}$.

For part (ii), let $\overline{A}$ and $\overline{B}$ denote the images of A and B in the quotient. By the induction hypothesis, there exists $\overline{x} \in \mathbf{U}/\, Z_{\mathbf{U}}(\mathcal{O})$ with $\overline{x} \equiv 1 \left(mod\, t^{k-(n-1)|m|+n}\right)$ such that $\overline{x} \cdot \overline{A} = \overline{B}$. We can now write $s(\overline{x}) \cdot A = ds\left(\overline{x} \cdot \overline{A}\right) + C$ and $B = ds\left(\overline{x} \cdot \overline{A}\right) + D$ for some $C, D \in \mathrm{Lie}(Z_{\mathbf{U}})_F$.

Notice that $s(\overline{x}) \equiv s(\overline{x})^{-1} \equiv 1 \;\left(mod\; t^{k-(n-1)|m|+n}\right)$. Therefore,

$$s(\overline{x}) \cdot A \;=\; s(\overline{x})\, A\, s(\overline{x})^{-1} \;+\; d\, s(\overline{x})\; s(\overline{x})^{-1} \;\equiv\; A \;\left(mod\; t^{k-n|m|+n}\right)$$

Since $A \equiv B \;\left(mod\; t^{k-n|m|+n}\right)$, it follows that $C \equiv D \;\left(mod\; t^{k-n|m|+n}\right)$. Now by the base case we can find $y \in Z_{\mathbf{U}}(\mathcal{O})$ with $y \equiv 1 \;\left(mod\; t^{k-n|m|+n+1}\right)$ such that $y \cdot C = D$. We conclude that $y\, s(\overline{x}) \cdot A = B$, because y is in the center. We clearly have $y\, s(\overline{x}) \equiv 1 \;\left(mod\; t^{k-n|m|+n+1}\right)$, as desired. □

3.4 Regular connections for solvable groups

We will fix a projection $\pi : \mathbf{k} \longrightarrow \mathbb{Q}$ as in the proof of Proposition 3.2. For $\mathbf{T}$ a torus, we extend this projection to a map $\pi : \mathrm{Lie}(\mathbf{T}) \cong X_*(\mathbf{T}) \otimes \mathbf{k} \longrightarrow X_*(\mathbf{T}) \otimes \mathbb{Q}$.

Proposition 3.26. *Let $\mathbf{G}$ be of the form $\mathbf{T} \ltimes \mathbf{U}$, where $\mathbf{T}$ is a torus and $\mathbf{U}$ is unipotent. Let $A = A_{\mathbf{T}} + A_{\mathbf{U}}$ be a formal connection with $A_{\mathbf{T}} \in Lie(\mathbf{T})_F$ a connection of the first kind and $A_{\mathbf{U}} \in Lie(\mathbf{U})_F$. Let b be a positive integer such that $b\, \pi\left((A_{\mathbf{T}})_{-1} \right) \in X_*(\mathbf{T})$. Then there exists $x \in \mathbf{G}(F_b)$ such that $x \cdot A = t^{-1} C_{\mathbf{T}} + t^{-1} C_{\mathbf{U}}$ for some $C_{\mathbf{T}} \in Lie(\mathbf{T})$ and $C_{\mathbf{U}} \in Lie(\mathbf{U})$. Moreover, we can arrange that $\pi(C_{\mathbf{T}}) = 0$ and $[C_{\mathbf{T}}, C_{\mathbf{U}}] = 0$.*

Proof. By the proof of Proposition 3.12, we can find $g \in \mathbf{T}(F)$ with $g \cdot A_{\mathbf{T}} = t^{-1}\, (A_{\mathbf{T}})_{-1}$. Set $\mu := b\, \pi\left((A_{\mathbf{T}})_{-1} \right) \in X_*(\mathbf{T})$. Then we have $(t^{\frac{1}{b}\mu}\, g) \cdot A_{\mathbf{T}} = t^{-1}\, C_{\mathbf{T}}$ for some $C_{\mathbf{T}} \in \mathrm{Lie}(\mathbf{T})$ with $\pi(C_{\mathbf{T}}) = 0$.

We can replace A with $B := (t^{\frac{1}{b}\, \mu}\, g) \cdot A$. We know that $B = t^{-1}\, C_{\mathbf{T}} + B_{\mathbf{U}}$ for some $B_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})_{F_b}$. By lifting to the b-ramified cover, we can assume that $B_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})_F$. We claim that we can find $u \in \mathbf{U}(F)$ such that $u \cdot B = t^{-1}\, C_{\mathbf{T}} + t^{-1}\, C_{\mathbf{U}}$ with $C_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})$ and $[C_{\mathbf{T}}, C_{\mathbf{U}}] = 0$. We will show this by induction on the dimension of $\mathbf{U}$.

The base case is $\mathbf{U} = \mathbb{G}_a$. Then, $\mathbf{T}$ acts on $\mathbf{U}$ by a character $\chi : \mathbf{T} \longrightarrow \mathbb{G}_m$. Write $B_{\mathbf{U}} = \sum_{j=r}^{\infty} (B_{\mathbf{U}})_j\, t^j$. For any $u = \sum_{j=r}^{\infty} u_j\, t^j \in \mathbf{U}(F)$, we have

$$u \cdot B = t^{-1}\, C_{\mathbf{T}} \,+\, B_{\mathbf{U}} \,-\, \sum_{j=r}^{\infty} \left(d\chi(C_{\mathbf{T}}) - j\right) u_j\, t^{j-1}$$

Since $\pi(C_{\mathbf{T}}) = 0$, we have $\pi\left(d\chi(C_{\mathbf{T}})\right) = 0$. There are two options:

(1) $d\chi(C_{\mathbf{T}}) \notin \mathbb{Q}$. Then, setting $u_j = \frac{1}{d\chi(C_{\mathbf{T}}) - j}\, (B_{\mathbf{U}})_{j-1}$ we get $u \cdot B = t^{-1}\, C_{\mathbf{T}}$.

(2) $d\chi(C_{\mathbf{T}}) = 0$. We can set $u_j = \frac{1}{d\chi(C_{\mathbf{T}}) - j}\, (B_{\mathbf{U}})_{j-1}$ for $j \neq 0$ and $u_0 = 0$. Then $u \cdot B = t^{-1}\, C_{\mathbf{T}} + t^{-1}\, (B_{\mathbf{U}})_{-1}$. Notice that we have $[C_{\mathbf{T}},\, (B_{\mathbf{U}})_{-1}] = d\chi(C_{\mathbf{T}})\, (B_{\mathbf{U}})_{-1} = 0$.

The base case follows.

Let's proceed with the induction step. We can decompose the action of the split torus $\mathbf{T}$ on the vector space $Z_{\mathbf{U}}$ into one-dimensional spaces. Let $\mathbf{H} \cong \mathbb{G}_a \leq Z_{\mathbf{U}}$

be one of these eigenspaces. The eigenspace decomposition yields a natural $\mathbf{T}$-equivariant section of the quotient map $Z_{\mathbf{U}} \longrightarrow Z_{\mathbf{U}}/\,\mathbf{H}$. We claim that we can extend this to a $\mathbf{T}$-equivariant section s of the morphism of schemes $\mathbf{U} \longrightarrow \mathbf{U}/\,\mathbf{H}$. In order to see this claim, we can use induction on the nilpotency class to reduce to the case when $\mathbf{U}$ has nilpotency class 1. Notice that we can find a section which is not necessarily $\mathbf{T}$-equivariant, since everything is isomorphic to an affine space. Then we can use the argument in Lemma 9.4 of [BS68] to obtain a $\mathbf{T}$-equivariant section. We can arrange so that s preserves the identities by substracting the image $s(1_{\mathbf{U}/\,\mathbf{H}})$. Let us denote by ds the induced map of tangent spaces at the identity.

Let $\overline{B}$ be the image of B in the quotient $\mathrm{Lie}(\mathbf{T} \ltimes \mathbf{U}/\,\mathbf{H})_F$. By the induction hypothesis, we can find $\overline{u} \in \mathbf{U}/\,\mathbf{H}(F)$ such that $\overline{u} \cdot \overline{B} = t^{-1}\, C_{\mathbf{T}} + t^{-1}\, \overline{D}$ for some $\overline{D} \in \mathrm{Lie}\,(\mathbf{U}/\,\mathbf{H})$ with $[C_{\mathbf{T}}, \overline{D}] = 0$. We can then write

$$s(\overline{u}) \cdot B = t^{-1}\, C_{\mathbf{T}} + t^{-1}\, ds(\overline{D}) + B_{\mathbf{H}}$$

for some $B_{\mathbf{H}} \in \mathrm{Lie}(\mathbf{H})_F$. Since s is $\mathbf{T}$-equivariant, we have $[ds(\overline{D}),\, C_{\mathbf{T}}] = 0$. We can now use the base case for $\mathbf{H}$ in order to conclude. □

Remark 3.27. *We can decompose the Lie algebra* $\mathfrak{u} := \mathit{Lie}(\mathbf{U})$ *into weight spaces* $\mathfrak{u} = \bigoplus_{i=1} \mathfrak{u}_{\chi_i}$. *Here each* χ_i *is a character of* $\mathbf{T}$. *Fix a basis* $\{\alpha_j\}$ *for the character lattice* $X^*(\mathbf{T})$. *For each* i *we can write* $\chi_i = \sum_j m^i_j \alpha_j$ *for some integers* m^i_j. *Define* $hgt(\chi_i) = \sum_j |m^i_j|$. *Set* $b = \max\limits_{1 \leq i \leq l} hgt(\chi_i)$. *If we don't require* $\pi(C_{\mathbf{T}})$ *and* $[C_{\mathbf{T}}, C_{\mathbf{U}}] = 0$ *in Proposition 3.26, then it suffices to pass to a* b*-ramified cover. So there is a uniform upper bound on the ramification needed to put any regular* $\mathbf{G}$*-connection into canonical form. It only depends on the solvable group* $\mathbf{G}$.

Let us prove a uniqueness result for regular canonical forms in the solvable case.

Proposition 3.28. *Let* $\mathbf{G}$ *be of the form* $\mathbf{T} \ltimes \mathbf{U}$ *as above. Let* $C = t^{-1}\, C_{\mathbf{T}} + t^{-1}\, C_{\mathbf{U}}$ *and* $D = t^{-1}\, D_{\mathbf{T}} + t^{-1}\, D_{\mathbf{U}}$ *be two regular canonical connections with* $C_{\mathbf{T}}, D_{\mathbf{T}} \in \mathit{Lie}(\mathbf{T})$ *and* $C_{\mathbf{U}}, D_{\mathbf{U}} \in \mathit{Lie}(\mathbf{U})$. *Suppose that* $\pi(C_{\mathbf{T}}) = \pi(D_{\mathbf{T}}) = 0$ *and* $[C_{\mathbf{T}}, C_{\mathbf{U}}] = [D_{\mathbf{T}}, D_{\mathbf{U}}] = 0$. *If there exists* $x \in \mathbf{G}(\overline{F})$ *such that* $x \cdot C = D$, *then in fact* $C_{\mathbf{T}} = D_{\mathbf{T}}$. *Moreover,* x *is in the centralizer* $Z_{\mathbf{G}}(C_{\mathbf{T}})(\mathbf{k})$ *of* $C_{\mathbf{T}}$.

Proof. By lifting to a ramified cover, we can assume that $x \in \mathbf{G}(F)$. Write $x = x_{\mathbf{U}}\, x_{\mathbf{T}}$ with $x_{\mathbf{U}} \in \mathbf{U}(F)$ and $x_{\mathbf{T}} \in \mathbf{T}(F)$. By the computation in Proposition 3.14 applied to $\mathbf{T}$, we get that $x_{\mathbf{T}} \in \mathbf{T}(\mathbf{k})$ and $C_{\mathbf{T}} = D_{\mathbf{T}}$. The same proof of Proposition 3.11 implies that $x \in \mathbf{G}(\mathbf{k})$ and $\mathrm{Ad}(x) C_{\mathbf{T}} = D_{\mathbf{T}}$. Since $C_{\mathbf{T}} = D_{\mathbf{T}}$, this means that $x \in Z_{\mathbf{G}}(C_{\mathbf{T}})(\mathbf{k})$. □

We conclude this section with a determinacy result for regular connections in the case of solvable groups. It follows from an analysis of the proof of Proposition 3.26. First we need to setup some notation.

Let $\mathbf{G} = \mathbf{T} \ltimes \mathbf{U}$ solvable. We have an action of the split torus $\mathbf{T}$ on the Lie algebra $\mathfrak{u} := \mathrm{Lie}(\mathbf{U})$ via the adjoint representation. We can decompose this representation into weight spaces $\mathfrak{u} = \bigoplus_{i=1}^{l} \mathfrak{u}_{\chi_i}$ for some finite set $\{\chi_1, \chi_2, ..., \chi_l\}$ of charaters $\chi_i : \mathbf{T} \longrightarrow \mathbb{G}_m$.

Suppose that we have a formal connection $A = A^{\mathbf{T}} + A^{\mathbf{U}}$, with $A^{\mathbf{T}} \in \text{Lie}(\mathbf{T})_F$ a connection of the first kind and $A^{\mathbf{U}} \in \text{Lie}(\mathbf{U})_F$. We can write $A^{\mathbf{T}} = t^{-1}A^{\mathbf{T}}_{-1} + \sum_{j=p}^{\infty} A^{\mathbf{T}}_j t^j$ for some $p \geq 0$ and $A^{\mathbf{U}} = \sum_{j=m}^{\infty} A^{\mathbf{U}}_j t^j$ for some $m \in \mathbb{Z}$. Let b be a positive integer such that $\mu := b\,\pi\left(A^{\mathbf{T}}_{-1}\right)$ is in $X_*(\mathbf{T})$. Define L to be

$$L := \max\left(\left\{\frac{1}{b}\langle \mu,\, \chi_i\rangle\right\}_{i=1}^{l} \cup \{0\}\right)$$

Proposition 3.29. *Keep the same notation as above. Assume that* $\mathbf{U}$ *has nilpotency class* n.

(i) Suppose that $m > L-1$. *Then there exists* $x \in \mathbf{G}(\mathcal{O})$ *with* $x \cdot A = t^{-1}A^{\mathbf{T}}_{-1}$. *More precisely, there exists* $x_{\mathbf{T}} \in \mathbf{T}(\mathcal{O})$ *with* $x_{\mathbf{T}} \equiv 1_{\mathbf{T}}$ $(mod\, t^{p+1})$ *and* $x_{\mathbf{U}} \in \mathbf{U}(\mathcal{O})$ *with* $x_{\mathbf{U}} \equiv 1_{\mathbf{U}}$ $(mod\, t^{m+1})$ *such that* $(x_{\mathbf{U}}x_{\mathbf{T}}) \cdot A = t^{-1}A^{\mathbf{T}}_{-1}$.

(ii) Suppose that $m \leq L-1$. *The* $\mathbf{G}(F)$*-gauge equivalence class of* A *is determined by the coefficients* $A^{\mathbf{T}}_j$ *for* $-1 \leq j < (n+1)(|m|-1)+L$ *and* $A^{\mathbf{U}}_j$ *for* $m \leq j < n(|m|-1)+L$. *More precisely, suppose that there is another connection* B *and positive integer* $k \geq n(|m|-1)+L$ *with* $B^{\mathbf{T}} \equiv A^{\mathbf{T}}$ $\left(mod\, t^{k+|m|}\right)$ *and* $B^{\mathbf{U}} \equiv A^{\mathbf{U}}$ $\left(mod\, t^{k}\right)$. *Then, there exists* $x \in \mathbf{G}(\mathcal{O})$ *with* $x \equiv 1$ $\left(mod\, t^{k-n|m|+n+1}\right)$ *such that* $x \cdot A = B$.

Proof. (i) By assumption $A^{\mathbf{T}} \equiv t^{-1}A^{\mathbf{T}}_{-1}$ $(mod\, t^{p})$. The proof of Proposition 3.12 shows that there exists $x_{\mathbf{T}} \in \mathbf{T}(\mathcal{O})$ with $x_{\mathbf{T}} \equiv 1$ $\left(mod\, t^{k+1}\right)$ such that $x_{\mathbf{T}} \cdot A^{\mathbf{T}} = t^{-1}A^{\mathbf{T}}_{-1}$. Set $C := x_{\mathbf{T}} \cdot A$. We can write $C = t^{-1}A^{\mathbf{T}}_{-1} + \text{Ad}(x_{\mathbf{T}})A^{\mathbf{U}}$. In order to ease notation, set $C^{\mathbf{U}} := \text{Ad}(x_{\mathbf{T}})A^{\mathbf{U}}$. Since $A^{\mathbf{U}} \equiv 0$ $(mod\, t^{m})$, we have $C^{\mathbf{U}} \equiv 0$ $(mod\, t^{m})$. We claim that there exists $x \in \mathbf{U}(\mathcal{O})$ with $x \equiv 1_{\mathbf{U}}$ $(mod\, t^{m+1})$ such that $x \cdot C = t^{-1}A^{\mathbf{T}}_{-1}$. This claim finishes the proof of part (i).

In order to prove the claim, we will induct on the nilpotency class of $\mathbf{U}$. The base case $n = 0$ means that $\mathbf{U} \cong \mathbb{G}_a^d$ for some d. We can decompose into eigenvalues and look at each coordinate separately in order to reduce to the case $d = 1$. Then there is a single weight space $\mathfrak{u}_{\chi_i}$. This case amounts to solving a recurrence as in the base case for the proof of Proposition 3.26. We want to find $x = \sum_{j=0}^{\infty} t^j u_j$ satisfying

$$C^{\mathbf{U}}_{j-1} = \left(d\chi_i(A^{\mathbf{T}}_{-1}) - j\right) u_j$$

If $j \leq m$ then $C^{\mathbf{U}}_{j-1} = 0$ by assumption. So we can set $u_j = 0$. If $j \geq m+1$, then we have

$$\pi\left(d\chi_i(A^{\mathbf{T}}_{-1})\right) - j = \frac{1}{b}\langle \mu, \chi_i\rangle - j \leq L - m - 1$$

By assumption $L - m - 1 < 0$, so we must have $d\chi_i(A^{\mathbf{T}}_{-1}) - j \neq 0$. Hence we can set $u_j = \frac{1}{d\chi_i(A^{\mathbf{T}}_{-1}) - j}\, C^{\mathbf{U}}_{j-1}$. The base case follows.

For the induction step, let $Z_{\mathbf{U}}$ denote the center of $\mathbf{U}$. Let s be a $\mathbf{T}$-equivariant section of the quotient $\mathbf{U} \longrightarrow \mathbf{U}/\,Z_{\mathbf{U}}$, as in the proof of Proposition 3.26. Let $\overline{C}$ be the image of C in the quotient $\mathrm{Lie}(\mathbf{T} \ltimes \mathbf{U}/\,Z_{\mathbf{U}})_F$. By the induction hypothesis, there exists $\overline{x} \in \mathbf{U}/\,Z_{\mathbf{U}}(\mathcal{O})$ such that $\overline{x} \equiv 1 \;\; (mod\, t^{m+1})$ and $\overline{x} \cdot \overline{C} = t^{-1}A^{\mathbf{T}}_{-1}$. We must then have $s(\overline{x}) \cdot C = t^{-1}A^{\mathbf{T}}_{-1} + D_{Z_{\mathbf{U}}}$ for some $D_{Z_{\mathbf{U}}} \in \mathrm{Lie}(Z_{\mathbf{U}})_F$. By definition

$$s(\overline{x}) \cdot C \;=\; t^{-1}\mathrm{Ad}(s(\overline{x}))A^{\mathbf{T}}_{-1} \;+\; \mathrm{Ad}(s(\overline{x}))C^{\mathbf{U}} \;+\; ds(\overline{x})s(\overline{x})^{-1}$$

We know that $s(\overline{x}) \equiv s(\overline{x})^{-1} \equiv 1 \;\; (mod\, t^{m+1})$. Also by assumption $C^{\mathbf{U}} \in \mathfrak{u}_{\mathcal{O}}$. It follows that

$$s(\overline{x}) \cdot C \;\equiv\; t^{-1}A^{\mathbf{T}}_{-1} \,+\, C^{\mathbf{U}} \;\equiv\; t^{-1}C_{\mathbf{T}} \;\; (mod\, t^{m})$$

Therefore $D_{Z_{\mathbf{U}}} \equiv 0 \;\; (mod\, t^{m})$. Now we can conclude by using the base case for $Z_{\mathbf{U}}$.

(ii) The hypothesis implies that $B^{\mathbf{T}}_{-1} = A^{\mathbf{T}}_{-1}$. The proof of Proposition 3.12 shows that there exist $x_{\mathbf{T}} \in \mathbf{T}(\mathcal{O})$ with $x_{\mathbf{T}} \equiv 1 \; \big(mod\, t^{k+|m|}\big)$ such that $x_{\mathbf{T}} \cdot A^{\mathbf{T}} = B^{\mathbf{T}}$. Set $C := x_{\mathbf{T}} \cdot A$. We have $C = B^{\mathbf{T}} + \mathrm{Ad}(x_{\mathbf{T}})A^{\mathbf{U}}$. Define $C^{\mathbf{U}} := \mathrm{Ad}(x_{\mathbf{T}})A^{\mathbf{U}}$. We know that $C^{\mathbf{U}} \equiv A^{\mathbf{U}} \; \big(mod\, t^{k}\big)$, because $x_{\mathbf{T}} \equiv 1 \; \big(mod\, t^{k+|m|}\big)$ and $A^{\mathbf{U}} \in t^{m}\mathfrak{u}_{\mathcal{O}}$. Therefore $C^{\mathbf{U}} \equiv B^{\mathbf{U}} \; \big(mod\, t^{k}\big)$ by assumption. We claim that there exists $u \in \mathbf{U}(\mathcal{O})$ with $u \equiv 1 \; \big(mod\, t^{k-n|m|+n+1}\big)$ such that $u \cdot C = B$. This claim concludes the proof of part (ii). In order to prove the claim, we will induct on the nilpotency class of $\mathbf{U}$. The base case $n = 0$ follows from an argument similar to the one for part (i), we ommit the details.

For the induction step, let $Z_{\mathbf{U}}$ and s be as in part (i). Let $\overline{C}$ and $\overline{B}$ denote the images of C and B in the quotient $\mathrm{Lie}(\mathbf{T} \ltimes \mathbf{U}/\,Z_{\mathbf{U}})_F$. By the induction hypothesis, there exists $\overline{x} \in \mathbf{U}/\,Z_{\mathbf{U}}(\mathcal{O})$ with $\overline{x} \equiv 1 \; \big(mod\, t^{k-(n-1)|m|+n}\big)$ such that $\overline{x} \cdot \overline{C} = \overline{B}$. We can now write $s(\overline{x}) \cdot C = ds\left(\overline{B}\right) + E_{Z_{\mathbf{U}}}$ and $B = ds\left(\overline{B}\right) + K_{Z_{\mathbf{U}}}$ for some $E_{Z_{\mathbf{U}}}, F_{Z_{\mathbf{U}}} \in \mathrm{Lie}(Z_{\mathbf{U}})_{F_b}$. By definition

$$s(\overline{x}) \cdot C \;=\; t^{-1}\mathrm{Ad}(s(\overline{x}))B^{\mathbf{T}} \;+\; \mathrm{Ad}(s(\overline{x}))C^{\mathbf{U}} \;+\; ds(\overline{x})s(\overline{x})^{-1}$$

We know that $s(\overline{x}) \equiv s(\overline{x})^{-1} \equiv 1 \; \big(mod\, t^{k-(n-1)|m|+n}\big)$. Since $|m| \geq 1$, we conclude that

$$ds\left(\overline{B}\right) + E_{Z_{\mathbf{U}}} \;=\; s(\overline{x}) \cdot C \;\equiv\; B^{\mathbf{T}} \,+\, C^{\mathbf{U}} \;=\; C \;\; \big(mod\, t^{k-n|m|+n}\big)$$

Since $k \geq k - n|m| + n$, we have $C \equiv B \; \big(mod\, t^{k-n|m|+n}\big)$. It follows that $E_{Z_{\mathbf{U}}} \equiv K_{Z_{\mathbf{U}}} \; \big(mod\, t^{k-n|m|+n}\big)$. Now by the base case we can find $y \in Z_{\mathbf{U}}(\mathcal{O})$ with $y \equiv 1 \; \big(mod\, t^{k-n|m|+n+1}\big)$ such that $(y\, s(\overline{x})) \cdot C = B$. We have $y\, s(\overline{x}) \equiv 1 \; \big(mod\, t^{k-n|m|+n+1}\big)$, as desired.

$\square$

Remark 3.30. *Suppose that* $\langle \mu, \chi_i \rangle > 0$ *for all* i. *It follows from the proof above that we can relax further the conditions on the coefficients of* $A^{\mathbf{U}}$. *Similarly, we can obtain sharper conditions for the coefficients of* $A^{\mathbf{T}}$ *in the case* $0 \leq m \leq L - 1$. *We leave the details of these refinements to the interested reader.*

Remark 3.31. *If* $L = 0$, *then the statement simplifies and we recover conditions similar to the unipotent case (Proposition 3.25).*

3.5 Regular connections for arbitrary linear algebraic groups

Theorem 3.32. *Let* $\mathbf{G}$ *be a connected linear algebraic group. Let* $A \in \mathfrak{g}_F$ *be a regular connection. Fix a Levi subgroup* $\mathbf{L}$ *and maximal torus* $\mathbf{T} \subset \mathbf{L}$*. Then there exists* $x \in \mathbf{G}(\overline{F})$ *such that* $x \cdot A = t^{-1}C$ *for some* $C \in \mathfrak{g}$*. Moreover, such* x *can be chosen so that the semisimple part* C_s *of the Levi component satisfies* $C_s \in \mathfrak{D}$ *and* $[C_s, C] = 0$*.*

Proof. Assume that A is of the first kind. Let $\mathbf{U} \subset \mathbf{G}$ be the unipotent radical of $\mathbf{G}$ with Lie algebra $\mathfrak{u}$. Let $\mathfrak{l}$ be the Lie algebra of $\mathbf{L}$. We know that $\mathbf{G} = \mathbf{L} \ltimes \mathbf{U}$, and so $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$. Decompose $A = A_{\mathfrak{l}} + A_{\mathfrak{u}}$. By the reductive group case, there exists $x \in \mathbf{L}(\overline{F})$ such that $x \cdot A_{\mathfrak{l}} = t^{-1}C$ for some $C \in \mathfrak{l}$ satisfying $C_s \in \mathfrak{D}$.

Let $C_n \in \mathfrak{l}$ denote the nilpotent part of C. Let $\mathbf{E}$ be the neutral component of the centralizer $Z_{\mathbf{T}}(C_n)$ of C_n in $\mathbf{T}$. Note that $\mathbf{E}$ is a subtorus of $\mathbf{T}$ and $C_s \in \text{Lie}(\mathbf{E})$. Since $\text{char}(\mathbf{k}) = 0$, there is a unique connected one-dimensional unipotent subgroup $\mathbf{N}$ of $\mathbf{L}$ with $C_n \in \text{Lie}(\mathbf{N})$. We have that $x \cdot A$ is a formal connection for the solvable group $(\mathbf{E} \times \mathbf{N}) \ltimes \mathbf{U}$. Now the result follows from the solvable case (Proposition 3.26). □

Remark 3.33. *In the beginning of the proof above, let* X *denote the semisimple part of* $(A_{\mathfrak{l}})_{-1}$*. After conjugating by an element of* $\mathbf{L}(\mathbf{k})$*, we can suppose that* $X \in$ *Lie*$(\mathbf{T})$*. Let* b *be a positive integer such that* $\mu := b\,\pi(X)$ *is in* $X_*(\mathbf{T})$*. Then, we can take* $x \in G(F_b)$ *in the proof above. In order to see this we can first apply* $t^{-\frac{1}{b}\mu}$*. So we can assume that* $\pi(X) = 0$*. By the proofs of Theorem 3.2 and Proposition 3.26, it follows that we don't need any further ramification to put* A *into canonical form.*

Let us state the corresponding uniqueness result for regular connections in general.

Proposition 3.34. *Let* $\mathbf{G}$ *be a connected linear algebraic group. Fix a Levi subgroup* $\mathbf{L}$ *and maximal torus* $\mathbf{T} \subset \mathbf{L}$*. Let* $C, D \in \mathfrak{g}$*. Write* C_s, D_s *for the semisimple parts of the Levi components* $C_{\mathfrak{l}}, D_{\mathfrak{l}}$*. Assume that* $C_s, D_s \in \mathfrak{D}$ *and* $[C_s, C] = [D_s, D] = 0$*. Suppose that there exists* $x \in \mathbf{G}(\overline{F})$ *such that* $x \cdot (t^{-1}\,C) = t^{-1}\,D$*. Then, we have* $C_s = D_s$*. Moreover* x *is in the centralizer* $Z_{\mathbf{G}}(C_s)(\mathbf{k})$*.*

Proof. Write $x = x_{\mathbf{U}}\,x_{\mathbf{L}}$ with $x_{\mathbf{U}} \in \mathbf{U}(\overline{F})$ and $x_{\mathbf{L}} \in \mathbf{L}(\overline{F})$. By Corollary 3.17 applied to $\mathbf{L}$, we get that $x_{\mathbf{L}} \in Z_{\mathbf{L}}(C_s)(\mathbf{k})$ and $C_s = D_s$. The same proof as in Proposition 3.11 shows that $x \in \mathbf{G}(\mathbf{k})$ and $\text{Ad}(x)C_s = D_s$. Since $C_s = D_s$, we conclude that $x \in Z_{\mathbf{G}}(C_s)(\mathbf{k})$. □

Let $A \in \mathfrak{g}_F$ be a regular formal connection. Proposition 3.34 implies that we can define the semisimple $\overline{F}$-monodromy $m_A^s \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$ just as we did in Definition 3.18. The same reasoning as in the reductive case yields the following.

Corollary 3.35. *Fix* $m \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$*. Let* $\mathcal{N}_{Z_{\mathbf{G}}(m)}$ *denote the nilpotent cone in the Lie algebra of* $Z_{\mathbf{G}}(m)$*. There is a natural correspondence*

$$\{\textit{regular } A \in \mathfrak{g}_{\overline{F}} \textit{ with } m_A^s = m\}\,/\,\mathbf{G}(\overline{F}) \quad \longleftrightarrow \quad \mathcal{N}_{Z_{\mathbf{G}}(m)}/\,Z_{\mathbf{G}}(m)$$

Since $\mathbf{T}$ is a split torus, we have a weight decomposition $\mathfrak{g} = \bigoplus_{\chi \in V} \mathfrak{g}_\chi$ of $\mathfrak{g}$ under the adjoint action of $\mathbf{T}$. Here V is a set of characters of $\mathbf{T}$. Let W be the Weyl group of $\mathbf{L}$ with respect to $\mathbf{T}$. There is a natural action of W on V. For any subset S of the quotient $V/\,W$, we can define the set Q_S of elements in $(X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})\,/\,W$ of type S just as we did for reductive groups. Let $Z_{\mathbf{G}}(S)$ be the centralizer of any element in Q_S. The same reasoning as in the reductive case yields the following concrete parametrization.

Corollary 3.36. *There is a natural correspondence*

$$\{\text{regular formal connections}\}\,/\,\mathbf{G}(\overline{F}) \quad \longleftrightarrow \quad \bigsqcup_{S \subset V/\,W} Q_S \times \mathcal{N}_{Z_{\mathbf{G}}(S)}/\,Z_{\mathbf{G}}(S)$$

3.6 Descent for gauge equivalence classes: Galois cohomology

All of the theorems above give us a description of connections up to trivializations over $\operatorname{Spec} \overline{F}$. We would like to get a classification over $\operatorname{Spec} F$. This amounts to a problem in Galois cohomology.

We have an action of $\mathbf{G}(\overline{F})$ on $\mathfrak{g}_{\overline{F}}$ that is compatible with the action of the absolute Galois group $\operatorname{Gal}(F)$. Choose a regular connection in canonical form $B = t^{-1}C$ with $C_s \in \mathfrak{D}$ and $[C_s, C] = 0$. It is a direct consequence of Proposition 3.34 that the centralizer of B in $\mathbf{G}(\overline{F})$ is $Z_G(C) := Z_{\mathbf{G}}(C)(\mathbf{k})$. Therefore, we get an exact sequence of sheaves of sets over the etale site of $\operatorname{Spec} F$

$$1 \longrightarrow Z_G(C) \longrightarrow \mathbf{G} \longrightarrow \mathbf{G} \cdot B \longrightarrow 1$$

Here $Z_G(C)$ is the constant sheaf associated to the group of $\mathbf{k}$-points of the centralizer $Z_{\mathbf{G}}(C)$ of C. This yields a long exact sequence of pointed sets:

$$1 \longrightarrow Z_G(C) \longrightarrow \mathbf{G}(F) \longrightarrow \mathbf{G} \cdot B(F) \longrightarrow H^1_{\operatorname{Gal}(F)}(Z_G(C)) \longrightarrow H^1_{\operatorname{Gal}(F)}(\mathbf{G})$$

The theorems of Tsen and Springer mentioned in the preliminaries imply that the right-most Galois cohomology group vanishes. This means that the set of connections over $\operatorname{Spec} F$ that admit a trivialization over $\operatorname{Spec} \overline{F}$ with canonical form $t^{-1}C$ is in bijection with $H^1_{\operatorname{Gal}(F)}(Z_G(C))$. Since the action of $\operatorname{Gal}(F)$ on $Z_G(C)$ is trivial, $H^1_{\operatorname{Gal}(F)}(Z_G(C))$ is in (noncanonical) bijection with the set conjugacy classes of elements of finite order in $Z_G(C)$. Such bijection comes from the choice of a topological generator of $\operatorname{Gal}(F) \cong \hat{\mathbb{Z}}$. Such a generator corresponds to the compatible choice of a generator ω_b of μ_b for all positive integers b. Here is a summary of the classification we have obtained.

Proposition 3.37 (Regular Connections over D^*)**.** *Let $B = t^{-1}C$ be a regular canonical connection with $C_s \in \mathfrak{D}$ and $[C_s, C] = 0$. The set of $\mathbf{G}$-connections over $\operatorname{Spec}(F)$ that become gauge equivalent to B over $\operatorname{Spec}(\overline{F})$ is in (noncanonical) bijection with the set of conjugacy classes of elements of finite order in $Z_{\mathbf{G}}(C)(\mathbf{k})$ as described below.*

The correspondence goes as follows. Let $x \in Z_{\mathbf{G}}(C)(\mathbf{k})$ of order b. By the vanishing of $H^1_{\mathrm{Gal}(F)}(\mathbf{G})$, we can find an element $y \in \mathbf{G}(F_b)$ such that $\omega_b \cdot y = y\, x$. The connection associated to x will be $A = y \cdot (t^{-1}\, C) \ \in \mathfrak{g}_F$. Conversely, suppose that $A = y \cdot B$ is a connection in $\mathfrak{g}_F$ for some $y \in \mathbf{G}(F_b)$. We set $x := y^{-1}\ (\omega_b \cdot y)$.

Using the descriptions of regular $\mathbf{G}(\overline{F})$- gauge equivalence classes we have given previously, we can parametrize regular formal connections. Let (m, u) be a pair with $m \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})$ and u a nilpotent element in $\mathcal{N}_{Z_{\mathbf{G}}(m)}$. A cohomology cocycle as described above is given by an element t of finite order in the centralizer $Z_{\mathbf{G}}(m, u)$ of u in $Z_{\mathbf{G}}(m)$. Since $Z_{\mathbf{G}}(m)$ is connected, we can conjugate by an element of $Z_{\mathbf{G}}(m)$ in order to assume that the semisimple element t lies in $\mathbf{T} \subset Z_{\mathbf{G}}(m)$. It follows that the set of regular formal $\mathbf{G}$ connections over D^* is in natural correspondence with equivalence classes of triples (m, x, u), where

(i) $m \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q})$.

(ii) x is an element of finite order in $\mathbf{T}(\mathbf{k})$.

(iii) $u \in \mathcal{N}_{Z_{\mathbf{G}}(m)}$ with $\mathrm{Ad}(t)(u) = u$.

Two such triples are considered equivalent if they can be conjugated by an element of $\mathbf{G}(\mathbf{k})$.

Recall that there is a canonical isomorphism $\mathbf{T}(\mathbf{k}) \cong X_*(\mathbf{T}) \otimes \mathbf{k}^{\times}$. Under this identification, the set $\mathbf{T}(\mathbf{k})^{tor}$ of elements of finite order in $\mathbf{T}(\mathbf{k})$ correspond to $X_*(\mathbf{T}) \otimes \mu_\infty$. The compatible compatible of primitive roots of unity ω_b yields an isomorphism $\mu_\infty \cong \mathbb{Q}/\,\mathbb{Z}$. Hence we get an identification $\mathbf{T}(\mathbf{k})^{tor} \cong X_*(\mathbf{T}) \otimes \mathbb{Q}/\,\mathbb{Z}$. This means that the set of pairs $(m, x) \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q}) \times \mathbf{T}(\mathbf{k})^{tor}$ is in natural bijection with $X_*(\mathbf{T}) \otimes\ \mathbf{k}/\,\mathbb{Z}$.

For an element $v \in X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Z}$, we will let $Z_{\mathbf{G}}(v)$ denote the centralizer $Z_{\mathbf{G}}(m, x)$ of the corresponding pair $(m, x) \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Q}) \times \mathbf{T}(\mathbf{k})^{tor}$. Conjugate elements of $X_*(\mathbf{T}) \otimes \mathbf{k}/\,\mathbb{Z}$ yield isomorphic centralizers, so it makes sense to define $Z_{\mathbf{G}}(v)$ for $v \in (X_*(\mathbf{T}) \otimes\ \mathbf{k}/\,\mathbb{Z})/\,W$. We end up the following parametrization of regular formal connections.

Corollary 3.38. *There is a natural bijection between regular formal connections over D^* and pairs (v, O), where*

(i) $v \in (X_*(\mathbf{T}) \otimes\ \mathbf{k}/\,\mathbb{Z})/\,W$.

(ii) O *is a nilpotent orbit in* $\mathcal{N}_{Z_{\mathbf{G}}(v)}/\,Z_{\mathbf{G}}(v)$.

Definition 3.39. *Let A be a regular formal connection over D^*. We will denote by (m^s_A, m^n_A) the corresponding pair granted by Corollary 3.38. m^s_A (resp. m^n_A) is called the semisimple (resp. unipotent) monodromy of A.*

Example 3.40. *Suppose that $\mathbf{k} = \mathbb{C}$. The set of pairs (m^s_A, m^n_A) as above is in correspondence with the set conjugacy classes in $\mathbf{G}(\mathbb{C})$. For a representative $(C, U) \in Lie(\mathbf{T}) \times \mathcal{N}_{\mathbf{G}}$ of the pair (m^s_A, m^n_A), the corresponding element of $\mathbf{G}(\mathbb{C})$ is given by $exp(2\pi i C + U)$. This just the monodromy class of the regular formal connection.*

We can use the theorem in [Hum95] pg. 26 to give a description of $Z_{\mathbf{G}}(v)$. We can decompose $\mathfrak{g} = \text{Lie}(\mathbf{T}) \oplus \bigoplus_{i=1}^{l} \mathfrak{u}_{\chi_i}$, where each $\mathfrak{u}_i$ is a one dimensional eigenspace of $\mathbf{T}$ consisting of nilpotent elements. Suppose that $\mathbf{T}$ acts on $\mathfrak{u}_i$ through the character $\chi_i : \mathbf{T} \longrightarrow \mathbb{G}_m$. Since we are working in characteristic 0, each $\mathfrak{u}_i$ is the Lie algebra of a unique unipotent subgroup $\mathbf{U}_i$ isomorphic to $\mathbb{G}_a$. Let $v \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\mathbb{Z})$. For each character $\chi \in X^*(\mathbf{T})$ it makes sense to ask whether $\langle v, \chi \rangle \in \mathbb{Z}$, even though v is only defined up to an element of $X_*(\mathbf{T})$. The connected component of $Z_{\mathbf{G}}(v)$ is generated by $\mathbf{T}$ and those unipotent weight spaces $\mathbf{U}_i$ such that $\langle v, \chi_i \rangle \in \mathbb{Z}$. The full group $Z_{\mathbf{G}}(v)$ is generated by its neutral component and the reflections w_α in the Weyl group W of $\mathbf{L}$ corresponding to roots $\alpha \in \Phi$ such that $\langle v, \alpha \rangle \in \mathbb{Z}$.

In order to classify formal regular connections, it is convenient to group them depending on the type of their semisimple monodromy. For each $v \in (X_*(\mathbf{T}) \otimes \mathbf{k}/\mathbb{Z})/W$, we can define the type of v to be a subset $S \subset V/W$ just as we did when working over $\overline{F}$. For any $S \subset V/W$, let us denote by $P_S \subset (X_*(\mathbf{T}) \otimes \mathbf{k}/\mathbb{Z})/W$ the set of all element of type S. By the description given in the last paragraph, it follows that the isomorphism class of $Z_{\mathbf{G}}(v)$ is the same for all $v \in P_S$. We will denote this group by $Z_{\mathbf{G}}(S)_F$. We can now rewrite the last corollary.

Corollary 3.41. *There is a natural correspondence*

$$\{\textit{regular formal connections over } D^*\} \quad \longleftrightarrow \quad \bigsqcup_{S \subset V/W} P_S \times \mathcal{N}_{Z_{\mathbf{G}}(S)_F} / Z_{\mathbf{G}}(S)_F$$

4 Irregular connections for G reductive

4.1 Connections in canonical form

For the rest of this section we will assume that $\mathbf{G}$ is connected reductive. We start with a definition.

Definition 4.1 (Canonical form)**.** *A connection $B \in \mathfrak{g}_{\overline{F}}$ is said to be in canonical form if we have $B = \sum_{j=1}^{l} D_j \, t^{r_j} + t^{-1} C$, where*

(1) $r_j \in \mathbb{Q}$ for all j, and they satisfy satisfy $r_1 < r_2 < ..r_l < -1$.

(2) $D_j \neq 0$ is a semisimple element in $\mathfrak{g}$ for all j.

(3) $D_1, D_2, ..., C$ are pairwise commuting (the brackets vanish).

The r_j above are called the levels of the canonical form. The smallest of them r_1 is called the principal level. The initial sum $\sum_{j=1}^{l} D_j \, t^{r_j}$ is called the irregular part of the connection; we denote it by B_{irr}.

Remark 4.2. *The irregular part could be 0 if the summation is empty. We then recover the notion of canonical form for a regular connection.*

As the reader might expect, it turns out that we can put any connection into canonical form. This generalizes what we did in the case of regular connections. We prove it first in the case of reductive groups.

Theorem 4.3 (Reduction Theory for Reductive Groups)**.** *Let $\mathbf{G}$ be connected reductive and $A \in \mathfrak{g}_{\overline{F}}$. Then there exists $x \in \mathbf{G}(\overline{F})$ such that $x \cdot A = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}\, C$ is in canonical form.*

The argument proceeds by induction on $\dim \mathbf{G}$. The base case (when $\mathbf{G} = \mathbb{G}_m$) follows from the computation done in the proof of Proposition 3.12. We state this result for future reference.

Proposition 4.4. *Let $\mathbf{G}$ be a tori.*

(a) Let $A = \sum_{j=r}^{\infty} A_j\, t^j$ a formal connection in $\mathfrak{g}_F$. Then there exists $x \in \mathbf{G}(\mathcal{O})$ such that $x \cdot A = \sum_{j=r}^{-1} A_j\, t^j$. Moreover there is a unique such x with $x \equiv 1 \pmod{t}$.

(b) Let $B = \sum_{j=r}^{-1} B_j\, t^j$ and $C = \sum_{j=r}^{-1} C_j\, t^j$ be two connections in canonical form. Suppose that there exists $x \in \mathbf{G}(F)$ such that $x \cdot C = B$. Then $x = g\, t^{\mu}$ for some cocharacter $\mu \in X_*(\mathbf{G})$ and some $g \in \mathbf{G}(\mathbf{k})$. In this case, we have $B_j = C_j$ for all $r \leq j < -1$ and $C_{-1} = B_{-1} - \mu$.

Proof. This is the same computation as in Propositions 3.12 and 3.14. We omit the details. □

Remark 4.5. *In particular, we see that two canonical connections $B = \sum_{j=r}^{-1} B_j\, t^j$ and $C = \sum_{j=r}^{-1} C_j\, t^j$ for a torus are gauge equivalent over F if and only if $B_j = C_j$ for all $r \leq j < -1$ and $C_{-1} - B_{-1} \in X_*(\mathbf{G})$. By lifting, we conclude that they are equivalent over $\overline{F}$ if and only if $B_j = C_j$ for all $r \leq j < -1$ and $C_{-1} - B_{-1} \in X_*(\mathbf{G}) \otimes \mathbb{Q}$.*

Let us start with the argument for Theorem 4.3. By the structure theory of reductive groups, we know that $\mathbf{G}$ admits an isogeny from the product of its maximal central torus and its derived subgroup $\mathbf{G}_{der}$. By Proposition 4.4, we can deal with the central part. We may therefore assume that $\mathbf{G}$ is semisimple.

By lifting to a ramified cover, we can assume $A = \sum_{j=r}^{\infty} A_j\, t^j \in \mathfrak{g}_F$ with $A_r \neq 0$. If $r \geq -1$ we can use the theory for regular connections developed in Section 2. So we can assume $r < -1$. There are two substantially different possibilities: A_r could be nilpotent or not. The case when A_r is not nilpotent turns out to be the easiest; we do it first.

4.2 The case when A_r is not nilpotent

We need the following lemma.

Lemma 4.6. *Let $A = \sum_{j=r} A_j\, t^j$ a connection in $\mathfrak{g}_F$ with $r < -1$. Let $V = (A_r)_s$ be the semisimple part of A_r. Then, there exist $x \in \mathbf{G}(F)$ such that $x \cdot A$ is in $\mathfrak{g}_V(F)$.*

Proof. This is very similar to Lemma 3.4. We will build inductively a sequence $(B_j)_{j=1}^{\infty}$ of elements of $\mathfrak{g}$ such that the gauge transformation $x := \lim_{n\to\infty} \prod_{j=0}^{n-1} \exp(t^{n-j}\, B_{n-j})$ satisfies the conclusion of the lemma.

Suppose that we have chosen B_j for $j \leq k$ such that the connection $A^{(k)} = \sum_{l=r}^{\infty} A_l^{(k)} t^l$ defined by $A^{(k)} := \prod_{j=0}^{k-1} \exp(t^{n-j} B_{k-n}) \cdot A$ satisfies $A_l^{(k)} \in \mathfrak{g}_V$ for all $l \leq k+r$. The base case $k = 0$ is trivial. Notice that we will have $A_r^{(k)} = A_r$. Let's try to determine B_{k+1}.

Recall that $\exp(t^{k+1} B_{k+1}) \equiv 1 + t^{k+1} B_{k+1} \pmod{t^{k+2}}$. By an elementary matrix computation (choose an embedding of $\mathbf{G} \hookrightarrow \mathrm{GL}_\mathbf{n}$), one can see that

$$\exp(t^{k+1} B_{k+1}) \cdot A^{(k)} \equiv \sum_{l=r}^{k+r} A_l^{(k)} t^l + [A_{k+1+r}^{(k)} - ad(A_r) B_{k+1}]\, t^{k+1+r} \pmod{t^{k+2+r}}$$

Let $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ be the spectral decomposition of $ad(X) = (ad(A_r))_s$. We know by definition that the operator $ad(A_{-r})$ restricts to an automorphism of $\mathfrak{g}_\lambda$ for all $\lambda \neq 0$. In particular, we can choose $B_{k+1} \in \mathfrak{g}$ such that $A_{k+1-r}^{(k)} - ad(A_{-r}) B_{k+1}$ is in $\mathfrak{g}_0 = \mathfrak{g}_V$. By induction, we are done with the construction of the sequence $(B_j)_{j=1}^{\infty}$. It is easy to see by construction that the gauge transformation $x := \lim_{n \to \infty} \prod_{j=0}^{n-1} \exp(t^{n-j} B_{n-j})$ satisfies $x \cdot A \in \mathfrak{g}_V(F)$. □

Let us continue the proof of Theorem 4.3. Suppose that A_r is not nilpotent. Then the semisimple part $X = (A_r)_s$ is not 0. Since we are assuming that $\mathbf{G}$ is semisimple, the connected reductive centralizer $\mathbf{Z}_G(X)$ is a proper subgroup of $\mathbf{G}$. By Lemma 4.6 we can assume that $A \in \mathfrak{g}_V(F)$. We win by induction, because $\dim \mathbf{Z}_G(V) < \dim \mathbf{G}$.

Recall that the principal level is defined to be the order r_1 of a canonical form (see the paragraph after Definition 4.1). We can define the principal level of a connection A to be the principal level of any canonical connection equivalent to A. This is well defined by Lemma 5.4 in the next section. The inductive argument given above implies the following interesting fact.

Proposition 4.7. *Suppose that $A = \sum_{j=r}^{\infty} A_j t^j$ with $r < -1$ and A_r not nilpotent. Then r is the principal level of A.*

Proof. We induct on the dimension of the group. The base case $\mathbb{G}_m$ is clear by direct computation. Notice that in the proof above we have that A_r is still not nilpotent in the smaller group $\mathbf{Z}_G(V)$, since its semisimple part V is not 0. We can then conclude by induction. □

Remark 4.8. *As we will soon see, this is not necessarily the case when A_r is nilpotent. In the nilpotent case the principal level can be larger than r.*

4.3 The case when A_r is nilpotent and proof of Theorem 4.3

This is a more delicate case. For this section we will let $\mathbf{G}$ be semisimple, as we may assume for our proof of Theorem 4.3. Let's set up the notation we need. Suppose that we have $A = \sum_{j=r}^{\infty} A_j\, t^j$ with A_r nilpotent and $r < -1$. Let $(H, X, Y = A_r)$ be a Jacobson-Morozov $\mathfrak{sl}_2$-triple coming from an algebraic homomorphism $\Phi : \mathrm{SL}_2 \longrightarrow \mathbf{G}$. For an integer n, we will denote by t^{nH} the element t^{μ}, where μ is the natural composition $\mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{\Phi} \mathbf{G}$.

Lemma 4.9. *With notation as above, there exists $x \in \mathbf{G}(F)$ such that $x \cdot A = \sum_{j=r}^{\infty} B_j \, t^j$ satisfies*

(1) $B_r = A_r$.

(2) $B_j \in \mathfrak{g}_X$ for all $j > r$.

Proof. This is a carbon copy of the proof of Lemma 4.6. The only difference is that in the last paragraph we have to use the fact that the range of $ad(A_r)$ is complementary to $\mathfrak{g}_X$. This follows from the theory of representations of $\mathfrak{sl}_2$. We ommit the details. $\square$

By Lemma 4.9, we can assume that $A_j \in \mathfrak{g}_X$ for $j > r$. For the purposes of having an algorithm that works in finitely many steps, we won't actually use the full force of the lemma as an input for the next proposition. Instead, we will use a weaker hypothesis. Let $\Lambda := \Lambda\left(A_r\right)$ be as in Definition 2.13. We will henceforth suppose that $A_{r+m} \in \mathfrak{g}_X$ for $1 \leq m < \Lambda(|r| - 1)$.

Let $(Z_l)_{l=1}^{q}$ be a basis of eigenvectors of $ad(H)$ acting on $\mathfrak{g}_X$. This means that $\mathfrak{g}_X = \bigoplus_{l=1}^{q} \mathbf{k} \, Z_l$ and that there exist λ_l such that $[H, Z_l] = \lambda_l \, Z_l$. It turns out that the λ_ls are nonnegative integers, by the theory of representations of $\mathfrak{sl}_2$. By the assumption on A, we can write $A_{r+m} = \sum_{l=1}^{q} a_{r+m,\, l} \, Z_l$ for all $1 \leq m \leq \Lambda(|r| - 1)$ and some constants $a_{r+m,\, l} \in \mathbf{k}$.

Definition 4.10. *In the situation above, define $\delta = \delta(A)$ to be given by:*

$$\delta = inf \left\{ \frac{m}{\frac{1}{2}\lambda_l + 1} \;:\; 1 \leq m < \Lambda(|r| - 1), \;\; 1 \leq l \leq q, \;\; a_{r+m,\, l} \neq 0 \right\}$$

We set $\delta = \infty$ if $A_{r+m} = 0$ for all $1 \leq m < \Lambda$. We also define the set

$$P := \left\{ (m, l) \;:\; 1 \leq m < \Lambda(|r| - 1), \;\; 1 \leq l \leq q, \;\; a_{r+m,\, l} \neq 0, \;\; \frac{m}{\frac{1}{2}\lambda_l + 1} = \delta \right\}$$

In plain words, P is the set of pairs (m, l) of indices in the definition of δ where the infimum is actually achieved.

Remark 4.11. *By the definition of $\Lambda(A_r)$, it follows that the denominators appearing in the set defining δ are always less than Λ. This implies that there exists a positive integer $b \leq 2\Lambda - 1$ such that $b\delta \in \mathbb{Z}$. This fact will be used later to determine a bound for the ramification needed to put A in canonical form.*

The following proposition is one of the main steps in the argument in [BV83]. What we are going to achieve here is to get a step closer to canonical form by applying a transformation of the type t^{nH}. These elements are called shearing transformations. The statement and proof in the case of $\mathbf{GL_n}$ can be found in [BV83] pages 33-34. We have decided to include a detailed treatment of the general case for the convenience of the reader.

Proposition 4.12 (Main Proposition for the Induction Step). *Let the notation/set up be as discussed above.*

(C1) Suppose $|r| - 1 \leq \delta \leq \infty$. Let $\tilde{A}$ be the 2-lift of A. Then $B := t^{(r+1)H} \cdot \tilde{A}$ is of the first kind, and B_{-1} only depends on A_{r+m} for $0 \leq m \leq \Lambda(|r| - 1)$.

(C2) Suppose $0 < \delta < |r| - 1$. We know that $b\delta \in \mathbb{Z}$ for some $b \in \mathbb{N}$. Let $\tilde{A}$ be the 2b-lift of A. We have that $B := t^{-b\delta H} \cdot \tilde{A}$ has order $r' := 2br + 2b\delta + 2b - 1 < -1$. Moreover,

$$B_{r'} = 2bA_r + 2b \sum_{(m,l) \in P} a_{r+m,\,l}\, Z_l \;\neq\; 2bA_r$$

In particular we have that $B_{r'}$ is determined by A_{r+m} for $0 \leq m < \Lambda(|r| - 1)$. If $B_{r'}$ is nilpotent, then $\dim(G \cdot B_{r'}) > \dim(G \cdot A_r)$.

Proof. The computation is similar to the one we did in the proof of Theorem 3.2. Recall from the discussion in that proof that for all $W \in \mathfrak{g}_\beta$ we have

$$\mathrm{Ad}(t^{nH})\, W = t^{n\beta(H)} W \quad (*)$$

(C1) By using the definitions and expanding

$$\begin{aligned} t^{(r+1)H} \cdot \tilde{A} = \; & 2 \sum_{m=0}^{\Lambda(|r|-1)-1} \mathrm{Ad}(t^{(r+1)H}) A_{r+m}\, t^{2(r+m)+1} \\ & + \mathrm{Ad}(t^{(r+1)H}) A_{r+\Lambda(|r|-1)}\, t^{2(r+\Lambda(|r|-1))+1} \\ & + 2 \sum_{m=\Lambda(|r|-1)+1}^{\infty} \mathrm{Ad}(t^{(r+1)H}) A_{r+m}\, t^{2(r+m)+1} \\ & + \frac{d}{dt}(t^{(r+1)H})\, t^{-(r+1)H} \end{aligned}$$

The fourth summand is just $(r+1)Ht^{-1}$, which is of the first kind. We can see that the third summand is actually in $\mathfrak{g}(\mathcal{O})$ by using $(*)$ and the fact that $(r+1)\beta(H) \geq (2\Lambda - 2)(r+1)$ for all roots β. The same reasoning implies that the second summand is of the first kind. For the first summand, we can write $A_{r+m} = \sum_{l=1}^{q} a_{r+m,\,l}\, Z_l$. We can expand and use $(*)$ plus the definition of λ_l. We get that the first summand is:

$$2 \sum_{m=0}^{\Lambda(|r|-1)-1} \sum_{l=1}^{q} a_{r+m,\,l}\, Z_l\, t^{2(r+m)+1+(r+1)\lambda_l}$$

This expression is also of the first kind. This can be shown by doing some algebra with the exponents of t, keeping in mind the definition of δ and the fact that $\delta \geq |r| - 1$. The remark about B_{-1} follows plainly from the argument, because the third summand did not contribute to B_{-1}.

(C2) This is very similar to the first case. We expand:

$$t^{-b\delta H} \cdot \tilde{A} = \; 2b \sum_{m=0}^{\Lambda(|r|-1)-1} \mathrm{Ad}(t^{-b\delta H}) A_{r+m}\, t^{2b(r+m)+2b-1}$$

$$+ \; 2b \sum_{m=\Lambda(|r|-1)}^{\infty} \mathrm{Ad}(t^{-b\delta H}) A_{r+m}\, t^{2b(r+m)+2b-1}$$

$$+ \; \frac{d}{dt}\left(t^{(-b\delta)H}\right) t^{(b\delta)H}$$

The third summand is $-b\delta H t^{-1}$, which is of the first kind. We can therefore ignore the third summand. The bound $-b\delta\beta(H) \geq -2b\delta(\Lambda-1)$ and equation $(*)$ show that the order of the second summand is at least $r'+1 = 2br+2b\delta+2b$. The computation is almost the same as for the third summand in Case 1 above. For the first summand, we can again use $A_{r+m} = \sum_{l=1}^{q} a_{r+m,\,l}\, Z_l$ and expand using $(*)$ to get:

$$2b \sum_{m=0}^{\Lambda(|r|-1)-1} \sum_{l=1}^{q} a_{r+m,\,l}\, Z_l\, t^{2b(r+m)+2b-1-b\delta\lambda_l}$$

We are reduced to check that the exponent of t in the sum above has minimal value $r' = 2br + 2b\delta + 2b - 1$ exactly for the pairs (m, l) in P. This is an exercise in elementary algebra.

The claim about $B_{r'}$ follows from the argument, because the second summand does not contribute to $B_{r'}$. The claim about the increase of the dimension of nipotent orbits is a direct consequence of Proposition 2.7.

□

Remark 4.13. *The essential point here is the claim about the dimension of the orbit appearing at the end of the proposition. This guarantees that the process of applying shearing transformations eventually stops. Hence we are provided with a terminating algorithm. See the proof of Theorem 4.3 given below for details.*

Example 4.14. *Let's see how this works in the case of SL_2. Up to inner automorphism, we can assume that $A_r = Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.*

In this case $\Lambda = 2$, and $\mathfrak{g}_X = \mathbf{k}X$. So there is a single eigenvalue $\lambda = 2$. Our assumption just says that A is of the form

$$A = Y\, t^r + \sum_{m=1}^{2|r|-3} a_{r+m} X\, t^{r+m} + \textit{higher order terms}$$

We have that δ is $\frac{n}{2}$, where n is the smallest index such that $a_{r+n} \neq 0$. The set P only contains this index n. So in fact A can be written in the form

$$A = Y\, t^r + \sum_{m=n}^{2|r|-3} a_{r+m} X\, t^{r+m} + \textit{higher order terms}$$

There are two cases.

(C1) The first case is $n \geq 2(|r|-1)$. This just means that all a_i above are 0. Then we can use the change of trivialization $t^{\frac{r+1}{2}H} = \begin{bmatrix} t^{\frac{r+1}{2}} & 0 \\ 0 & t^{-\frac{r+1}{2}} \end{bmatrix}$ to transform A into a connection of the first kind.

(C2) The second case is when $n < 2(|r|-1)$. So at least some of the a_i are not 0. In this case we can apply the transformation $t^{-\frac{n}{4}H} = \begin{bmatrix} t^{-\frac{n}{4}} & 0 \\ 0 & t^{\frac{n}{4}} \end{bmatrix}$. The resulting connection will have order $r + \frac{n}{2}$. The principal term will be $B_{r+\frac{n}{2}} = Y + a_{r+n}X$, which is semisimple. Hence we can use Lemma 4.6 to reduce to the group $\mathbb{G}_m$. We can then apply Proposition 4.4 to find the canonical form.

Proof of Theorem 4.3. By Lemma 4.9, we can put ourselves in the situation of Proposition 4.12 above. We have three possibilities:

(i) If $|r| - 1 \leq \delta \leq \infty$, then we can use Proposition 4.12 Case 1. We are done by the theory of regular connections we have already developed.

(ii) If $0 < \delta < |r| - 1$, we can use 4.12 Case 2. Suppose that $B_{r'}$ is not nilpotent. Then we are in the case worked out in Subsection 4.2.

(iii) Suppose that $0 < \delta < |r| - 1$ and $B_{r'}$ is nilpotent with $\dim(G \cdot B_{r'}) > \dim(G \cdot A_r)$. We can apply Proposition 4.12 Case 2 again with B instead of A. We can keep iterating this procedure until we are in one of the first two possibilities above. Notice that this process cannot go on indefinitely, because the dimensions of nilpotent orbits in $\mathbf{G}$ are bounded.

$\square$

Remark 4.15. *The dimension of adjoint nilpotent orbits in $\mathbf{G}$ is always even [CM93]. Therefore we need to apply at most $\left\lfloor \frac{1}{2} dim(\mathbf{G}) \right\rfloor$ shearing transformations as in Proposition 4.12 Case 2 before we land in one of the first two possibilities.*

4.4 Algorithm for reductive groups and some quantitative results

Let us give a detailed description of the reduction algorithm that we obtain from the proof of Theorem 4.3.

Algorithm 4.16 (Algorithm for reduction of a formal connection for a reductive group)**.** *There is a set of six possible operations that we will use as steps in our algorithm.*

(i) Apply Lemma 4.6.

(ii) Apply Lemma 4.9.

(iii) Apply Proposition 4.12 Case 1.

(iv) Apply Proposition 4.12 Case 2.

(v) Find the canonical form of a connection in a torus.

(vi) Find the canonical form for a connection of the first kind in a semisimple group (as in Theorem 3.2 of Section 3).

The algorithm proceeds as follows. The input is a given reductive group $\mathbf{G}$ *and a formal connection* $A = \sum_{j=r}^{\infty} A_j\, t^j$*. First, we know that* $\mathbf{G}$ *is isogenous to the product* $Z^0(\mathbf{G}) \times \mathbf{G}_{der}$ *of its maximal central torus and its derived subgroup. Apply operation (v) to the central part of the connection* $A_{Z^0(\mathbf{G})}$*. We can record the (canonical form) output of (v) and ignore it from now on, since it is not going to be altered by the subsequent steps in the algorithm. Replace* $\mathbf{G}$ *by* $\mathbf{G}_{der}$ *and* A *by* A_{der}*. We have two cases.*

(1) If A_{der} *is of the first kind, apply step (vi) to reduce this connection to canonical form. Add any "central" parts we might have split off earlier in the algorithm and output the result. End of the algorithm.*

(2) If A_{der} *is not of the first kind, check whether* A_r *is nilpotent or not. There are now two ways to proceed:*

- *If* A_r *is not nilpotent, use operation (i). Replace* $\mathbf{G}$ *by* $Z_{\mathbf{G}}\left((A_r)_s\right)$ *and replace* A *by the output of operation (i). Return to the beginning of the algorithm with this new input.*
- *If* A_r *is nilpotent, compute* $\Lambda(A_r)$*. Apply operation (ii) and replace* A *with the output. Now compute* δ*.*
  - *(a) If* $|r| - 1 \leq \delta$*, apply operation (iii) and replace* A *with the output. This is a connection of the first kind. Return to the beginning of the algorithm.*
  - *(b) If* $\delta < |r| - 1$*, apply operation (iv). Go to the beginning of the algorithm.*

Remark 4.17. *In the algorithm above the order of the pole of* A *only ever gets smaller (taking into account b-lifting whenever passing to a ramified cover). This shows that the principal level of* A *determines the mildest pole in the* $\mathbf{G}(\overline{F})$*-gauge equivalence class of* A*.*

In each step of Algorithm 4.16 we are ultimately working with a semisimple subgroup of $\mathbf{G}$. These subgroups are of the form $\mathbf{H}_{der}$, where $\mathbf{H}$ is the centralizer of a finite set $\{D_1, D_2, ..., D_l\}$ of pairwise commuting semisimple elements in $\mathfrak{g}$.

A careful study of Algorithm 4.16 yields a bound for the ramification needed to put a given connection into canonical form. It turns out that we can get a uniform bound that only depends on the group, and not on the connection. Before giving the proof, we will need a lemma.

Lemma 4.18. *Let $\mathbf{G}$ be a connected semisimple group. Let R be the rank of $\mathbf{G}$. Suppose that $\{D_1, D_2, ..., D_l\}$ is a finite set of pairwise commuting semisimple elements in $\mathfrak{g}$. Set $\mathbf{H} = Z_{\mathbf{G}}(\{D_1, D_2, ..., D_l\})$, the centralizer of all D_i. Let $\mathbf{H}_{der}$ be the derived subgroup of $\mathbf{H}$. Then we have $J(\mathbf{H}_{der}) \leq hgt(\mathfrak{g})^{2R-2} \cdot J(\mathbf{G})$.*

Proof. The lemma is clearly true if $\mathbf{H} = \mathbf{G}$. We can therefore assume that $\mathbf{H} \neq \mathbf{G}$.

Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ such that $\mathrm{Lie}(\mathbf{T})$ contains the set $\{D_i\}$ of pairwise commuting semisimple elements. Note that $\mathbf{T} \subset \mathbf{H}$ is a maximal torus. Let Φ (resp. Σ) be the set of roots of $\mathbf{G}$ (resp. $\mathbf{H}$) with respect to $\mathbf{T}$. We will denote by $\Sigma^\vee$ and $\Phi^\vee$ the corresponding sets of coroots. It follows by definition that $\Sigma \subset \Phi$ and $\Sigma^\vee \subset \Phi^\vee$.

Write $Q_{\mathbf{H}_{der}}$ for the coweight lattice of $\mathbf{H}_{der}$. Let $\lambda \in Q_{\mathbf{H}_{der}}$. We want to show that there exists $b \leq \mathrm{hgt}(\mathfrak{g})^{2R-1} \cdot J(\mathbf{G}_{\mathrm{der}})$ such that $b\lambda \in \mathbb{Z}\Sigma^\vee$.

Fix a choice of positive roots Φ^+ in Φ. Let Δ_Φ be the corresponding set of simple roots. By definition $|\Delta_\Phi| = R$. Notice that this induces a set of positive roots $\Sigma^+ := \Phi^+ \cap \Sigma$. Let Δ_Σ be the corresponding set of simple roots in Σ. Set $c := |\Delta_\Sigma|$. We know that $c \leq R - 1$ because $\mathbf{H} \neq \mathbf{G}$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}\Sigma \xrightarrow{M} \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi/\,\mathbb{Z}\Sigma \longrightarrow 0$$

The theory of Smith normal form implies that $\mathbb{Z}\Phi/\,\mathbb{Z}\Sigma \cong E \oplus \mathbb{Z}^{R-c}$, where E is a finite group. The exponent of E is given by the biggest elementary divisor d of the inclusion M of free $\mathbb{Z}$-modules. Applying the functor $\mathrm{Hom}(-, \mathbb{Z})$ to the short exact sequence yields an exact sequence

$$0 \longrightarrow \mathbb{Z}^{R-c} \longrightarrow Q_{\mathbf{G}} \longrightarrow Q_{\mathbf{H}_{der}} \longrightarrow E \longrightarrow 0$$

Hence we have that $d\lambda$ can be extended to an element of $Q_{\mathbf{G}}$. By the definition of $J(\mathbf{G})$, it follows that $d\, J(\mathbf{G})\, \lambda$ extends to an element of $\mathbb{Z}\Phi^\vee$.

Let $\varphi : \mathbb{Z}\Phi^\vee \longrightarrow Q_{\mathbf{H}_{der}}$ be the composition

$$\varphi : \mathbb{Z}\Phi^\vee \hookrightarrow Q_{\mathbf{G}} \longrightarrow Q_{\mathbf{H}_{der}}$$

Set $L := \mathrm{Im}\, \varphi$ and $K := \mathrm{Ker}\, \varphi$. The discussion above implies that the exponent of the finite group $Q_{\mathbf{H}_{der}}/\, L$ is bounded by $d\, J(\mathbf{G})$. By definition we have a short exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}\Phi^\vee \longrightarrow L \longrightarrow 0$$

Since L is a torsion-free $\mathbb{Z}$-module, the above exact sequence splits. Fix a splitting $\mathbb{Z}\Phi^\vee \cong K \oplus L$. Let's look now at the inclusion of lattices $\mathbb{Z}\Sigma^\vee \subset \mathbb{Z}\Phi^\vee$. The composition

$$\mathbb{Z}\Sigma^\vee \hookrightarrow \mathbb{Z}\Phi^\vee = K \oplus L \xrightarrow{pr_2} L \hookrightarrow Q_{\mathbf{H}_{der}}$$

is the natural inclusion $\mathbb{Z}\Sigma^\vee \hookrightarrow Q_{\mathbf{H}_{der}}$. Hence the morphism ψ given by the composition

$$\psi : \mathbb{Z}\Sigma^\vee \hookrightarrow \mathbb{Z}\Phi^\vee = K \oplus L \xrightarrow{pr_2} L$$

is injective. So we have an inclusion $\psi : \mathbb{Z}\Sigma^\vee \hookrightarrow L$. Let e denote the exponent of the finite group $L/\,\mathbb{Z}\Sigma^\vee$. By definition e is the biggest elementary divisor of the inclusion $\psi : \mathbb{Z}\Sigma^\vee \hookrightarrow L$. Notice that this is also the biggest elementary divisor of the natural inclusion $\mathbb{Z}\Sigma^\vee \subset \mathbb{Z}\Phi^\vee = K \oplus L$. The discussion up to now implies that $J(\mathbf{H}_{der}) \leq e\, d\, J(\mathbf{G})$.

We are left to compute the elementary divisors d and e of the inclusions of the root and coroot lattices. We first claim that $d \leq \text{hgt}(\mathfrak{g})^{R-1}$. In order to prove the claim, we will use Δ_Σ and Δ_Φ as bases for the root lattices.

For each $\alpha \in \Delta_\Sigma$, we can write $\alpha = \sum_{\beta \in \Delta_\Phi} m_\beta^\alpha \beta$ for some nonnegative integers m_β^α. Set $M := (m_\beta^\alpha)_{\beta \in \Delta_\Phi,\, \alpha \in \Delta_\Sigma}$. This is a $R \times c$ matrix representing the inclusion $\mathbb{Z}\Sigma \hookrightarrow \mathbb{Z}\Phi$. By the theory of Smith normal form, d divides all $c \times c$-minors of M. Since all m_β^α are nonnegative, such $c \times c$-minor is bounded by

$$\prod_{\alpha \in \Delta_\Sigma} \left(\sum_{\beta \in \Delta_\Phi} m_\beta^\alpha \right) \leq \text{hgt}(\mathfrak{g})^c \leq \text{hgt}(\mathfrak{g})^{R-1}$$

The claim $d \leq \text{hgt}(\mathfrak{g})^{R-1}$ follows. We can apply the same argument to the inclusion $\mathbb{Z}\Sigma^\vee \subset \mathbb{Z}\Phi^\vee$. The maximal height in the dual root system $\Phi^\vee$ is also $\text{hgt}(\mathfrak{g})$. Therefore the same proof yields a bound $e \leq \text{hght}(\mathfrak{g})^{R-1}$. This implies

$$J(\mathbf{H}_{der}) \leq e\, d\, J(\mathbf{G}) \leq \text{hgt}(\mathfrak{g})^{2R-2} \cdot J(\mathbf{G})$$

$\square$

Proposition 4.19. *Let $\mathbf{G}$ connected reductive. Let $A \in \mathfrak{g}_F$ be a connection. Then there exist $x \in \mathbf{G}(F_b)$ for some positive integer b such that $x \cdot A$ is in canonical form. Let $R := rank(\mathbf{G}_{der})$. Then b can be chosen so that*

$$b \leq 2\, hgt(\mathfrak{g})^{2R-1} \cdot J(\mathbf{G}_{der}) \cdot \prod_{j=0}^{\left\lfloor \frac{dim(\mathbf{G}_{der})}{3} \right\rfloor} (4\, hgt(\mathfrak{g}) + 2)^{\left\lfloor \frac{1}{2}(dim(\mathbf{G}_{der}) - 3j) \right\rfloor}$$

Proof. We have to keep track of how much ramification is needed to perform each of the steps in Algorithm 4.16. Recall our six operations:

(i) Apply Lemma 4.6. No ramification is needed for this operation, as is apparent from the proof of the lemma.

(ii) Apply Lemma 4.9. No ramification is needed for this operation. This also follows directly from the proof of the lemma.

(iii) Apply Proposition 4.12 Case 1. We need to pass to a 2-cover.

(iv) Apply Proposition 4.12 Case 2. We need to pass to a $2b$-cover, where b is such that $b\delta \in \mathbb{Z}$. By Remark 4.11, we know that we can choose $b \leq 2\Lambda - 1 \leq 2\mathrm{hgt}(\mathfrak{g}) + 1$.

(v) Find the canonical form of a connection in a torus. No ramification is needed to perform this operation, by the proof of Proposition 3.12.

(vi) Find the canonical form for a connection of the first kind (as in Theorem 3.2). By Lemma 3.9, we can perform this operation after passing to a b-cover with $b \leq \mathrm{hgt}(\mathfrak{g}) \cdot I(\mathbf{G})$.

We know that operations (iii) and (vi) will be used only once, at the end of the algorithm. This gives us a factor of $2\,\mathrm{hgt}(\mathfrak{g}) \cdot I(\mathbf{H}_{\mathrm{der}})$, where $\mathbf{H}$ is the centralizer $Z_{\mathbf{G}_{der}}(\{D_1, D_2, ..., D_l\})$ of a finite set of pariwise commuting semisimple elements D_i in $\mathfrak{g}_{der}$. Since $I(\mathbf{H}_{der}) \leq J(\mathbf{H}_{der})$, this is bounded by $2\,\mathrm{hgt}(\mathfrak{g}) \cdot J(\mathbf{H}_{\mathrm{der}})$. By Lemma 4.18 we known that $J(\mathbf{H}_{\mathrm{der}}) \leq \mathrm{hgt}(\mathfrak{g})^{2R-2} \cdot J(\mathbf{G}_{\mathrm{der}})$. This yields the first factor in the bound above.

We are now left to count the amount of times that we need to apply operation (iv) in our algorithm. Each time we apply it we need to pass to a cover of ramification at most $4\,\mathrm{hgt}(\mathfrak{g}) + 2$. By the remark after the proof of Theorem 4.3, we need to apply operation (iv) at most $\left\lfloor \frac{1}{2}\mathrm{dim}(\mathbf{G}_{\mathrm{der}}) \right\rfloor$ times before we are in the case when A_r is not nilpotent. We therefore pick up a ramification of at most $(4\,\mathrm{hgt}(\mathfrak{g}) + 2)^{\left\lfloor \frac{1}{2}\mathrm{dim}(\mathbf{G}_{\mathrm{der}}) \right\rfloor}$. After that we change our group.

We can apply operation (i) and split off the central part in order to pass to a proper semisimple subgroup $\mathbf{H}_{der} := \left(Z_{\mathbf{G}}\left(\,(A_r)_s\,\right)\right)_{\mathrm{der}}$. Notice that $\mathrm{dim}(\mathbf{H}_{der}) \leq \mathrm{dim}(\mathbf{G}_{\mathrm{der}}) - 3$, because we are removing at least two root spaces (positive and negative pair) and the nontrivial central torus of the centralizer $\mathbf{H}$. Now we start all over again. We know that we need to apply operation (iv) at most $\left\lfloor \frac{1}{2}\left(\mathrm{dim}(\mathbf{G}_{\mathrm{der}}) - 3\right) \right\rfloor$-many times until A_r is not nilpotent. So we pick up a ramification of at most $(4\,\mathrm{hgt}(\mathfrak{g}) + 2)^{\left\lfloor \frac{1}{2}(\mathrm{dim}(\mathbf{G}_{\mathrm{der}}) - 3) \right\rfloor}$. Iterating this procedure, we get the product appearing in the bound above. $\square$

Remark 4.20. *In terms of dimension, the right hand side is* $J(\mathbf{G}_{der}) \cdot e^{O\left(dim(\mathbf{G}_{der})^2 \log dim(\mathbf{G}_{der})\right)}$.

We proceed to establish a quantitative refinement of Theorem 4.3. It essentially follows from keeping track of some indices in the operations for the algorithm above. It makes sense once we know that the irregular part of a canonical form is unique up to conjugacy, as stated in Theorem 5.5 below. It should be remarked that the statement of this result over $\mathbb{C}$ can already be found in the work of Babbitt and Varadarajan [BV83] page 76.

Proposition 4.21 (Determinacy for the Irregular part of the Canonical Form)**.** *Let $\mathbf{G}$ be connected reductive. Let $A = \sum_{j=r}^{\infty} A_j t^j$ be a connection in $\mathfrak{g}_F$. The irregular part of the canonical form of A depends only on A_{r+m} for $0 \leq m < (hgt(\mathfrak{g}) + 1)\,(|r| - 1)$.*

Proof. It suffices to check the steps in Algorithm 4.16. In some steps of the algorithm we replace the group $\mathbf{G}$ by a proper subgroup (either a centralizer or the derived subgroup). This can only decrease the quantity $(\mathrm{hgt}(\mathfrak{g})+1)(|r|-1)$, so we can safely ignore these changes of groups. We are left to study the effect of operations (i)-(vi) in Algorithm 4.16.

The last operation (vi) has no effect on the irregular part of the connection, so there is nothing to do here. Step (v) takes a connection $A = \sum_{j=r}^{\infty} A_j\, t^j$ and outputs its truncation $A = \sum_{j=r}^{-1} A_j\, t^j$ (see the proof of Proposition 3.12). The output is therefore determined by the coefficients given in the statement of the proposition.

Step (iii) outputs a connection with no irregular part. Notice that in Proposition 4.12 we can determine if we are in Case 1 (i.e. when we have to perform Step (iii)) based on the value of δ. This depends only on the A_{r+m} for $0 \leq m < \Lambda(A_r)(|r|-1)$. Since $\Lambda(A_r) \leq \mathrm{hgt}(\mathfrak{g})+1$ by Example 2.14, this case can be determined by the coefficients provided.

For the remaining operations (i), (ii) and (iv), we start with a given connection A with lowest coefficient A_r and output an irregular connection B with lowest coefficient $B_{r'}$. The proposition will follow if we can prove that for each of these operations the coefficients $B_{r'+m}$ for $0 \leq m < (\mathrm{hght}+1)(|r'|-1)$ are completely determined by the coefficients A_{r+m} for $0 \leq m < (\mathrm{hgt}(\mathfrak{g})+1)(|r|-1)$.

Operations (i) and (ii) are very similar. In this case we have $r = r'$. Let m be an integer. From the proofs of Lemma 4.6 and Lemma 4.9, it follows that B_m is determined by A_j for $j \leq m$. So we are done with these operations.

We are left with operation (iv). Recall from the proof of Proposition 4.12 Case 2 that we have

$$B = t^{-b\delta H} \cdot \tilde{A} = \;\; 2b \sum_{m=0}^{\infty} \mathrm{Ad}(t^{-b\delta H}) A_{r+m}\, t^{2b(r+m)+2b-1} \;-\; b\delta H t^{-1}$$

The term $b\delta H t^{-1}$ is determined by the knowledge of δ and $H = (A_r))_s$. For the infinite sum, we can write the root decompositions $A_{r+m} = \sum_{\beta\in\Phi} A^{\beta}_{r+m}$ and use that $\mathrm{Ad}(t^{-b\delta H})\, A^{\beta}_{r+m} = t^{-b\delta\beta(H)}\, A^{\beta}_{r+m}$ in order to get

$$\mathrm{Ad}(t^{-b\delta H}) A^{\beta}_{r+m}\, t^{2b(r+m)+2b-1} \; = A^{\beta}_{r+m}\, t^{-b\delta\beta(H)+2b(r+m)+2b-1} = A^{\beta}_{r+m}\, t^{r'+2bm-b\delta\beta(H)}$$

By Example 2.14 we know $\beta(H) \leq 2\,\mathrm{hgt}(\mathfrak{g})$. Suppose that a positive integer m satisfies $2bm - b\delta\beta(H) < (\mathrm{hgt}(\mathfrak{g})+1)(|r'|-1)$. Some algebraic manipulations show that $m < (\mathrm{hght}+1)(|r|-1)$. So indeed the coefficients $B_{r'+m}$ for $0 \leq m < (\mathrm{hght}+1)(|r'|-1)$ are completely determined by the coefficients A_{r+m} for $0 \leq m < (\mathrm{hgt}(\mathfrak{g})+1)(|r|-1)$. □

Remark 4.22. *Proposition 4.21 can be thought of as a continuity statement. It says that a small perturbation of the original connection will not alter the irregular part of its canonical form. This is analogous to the finite determinacy theorem for analytic singularities, as in [dJP00] Theorem 9.4 in page 313.*

One can obtain a similar continuity statement for the residue of the connection (i.e. the coefficient of t^{-1} in the canonical form). However the explicit bound for the number of terms needed is complicated and not very illuminating. We therefore do not include a formula for the bound.

Proposition 4.23. *Let* $\mathbf{G}$ *be connected reductive and let* $A \in \mathfrak{g}_F$ *be a connection. There exist a positive integer* n *such that all connections* $C \in \mathfrak{g}_F$ *satisfying* $C \equiv A \pmod{t^n}$ *are* $\mathbf{G}(\overline{F})$*-gauge equivalent to* A.

Proof. In Algorithm 4.16 we apply operations (iii) and (vi) exactly once at the very end.

Suppose that we are given the coefficients A_{r+m} for $0 \leq m \leq (\mathrm{hgt}(\mathfrak{g})+1)\,(|r|-1)$ in a given connection A. Let D be the output of applying one of the operations (i), (ii), (iv) or (v) to A. The proof of Proposition 4.21 implies that we can determine the corresponding coefficients $D_{r'+m}$ for $0 \leq m \leq (\mathrm{hgt}(\mathfrak{g})+1)\,(|r'|-1)$.

We can iterate this reasoning. Suppose that $D = \sum_{j=r'}^{\infty} D_j\, t^j$ is the ouput of the algorithm before applying the last two steps (operations (iii) and (vi)). Then we see that the coefficients $0 \leq m < (\mathrm{hgt}(\mathfrak{g})+1)\,(|r'|-1)$ are completely determined by A_{r+m} for $0 \leq m \leq (\mathrm{hgt}(\mathfrak{g})+1)\,(|r|-1)$, where A is the original connection we start with. The number of steps needed in the algorithm is also completely determined.

By the statement of Proposition 4.12 Case 1, we will be able to determine the residue (i.e. the coefficient of t^{-1}) when we apply operation (iii) to D. The output of operation (iii) will be a connection of the first kind $B = \sum_{j=-1}^{\infty} B_j\, t^j$, and we can compute $k\,(B_{-1})$ (see the remark after Lemma 3.4).

We now need to determine the result of applying operation (vi) to B as above. By Remark 3.5, this will be determined by B_j for $-1 \leq j \leq k(B_{-1})$. We can then work backwards using an argument similar to the proof of Proposition 4.21 to find a number n big enough so that the coefficients B_j for $-1 \leq j \leq k(B_{-1})$ are determined by A_j for $r \leq j < n$. □

5 Irregular connections for arbitrary linear algebraic groups

We will proceed as in the regular case in order to prove the existence canonical form of a connection for any connected linear algebraic group. Just as before, we start with the solvable case.

5.1 Irregular connections for solvable groups

We will again make use of the map $\pi : \mathrm{Lie}(\mathbf{T}) \cong X_*(\mathbf{T}) \otimes \mathbf{k} \longrightarrow X_*(\mathbf{T}) \otimes \mathbb{Q}$ as in Proposition 3.26.

Proposition 5.1. *Let* $\mathbf{G}$ *be of the form* $\mathbf{T} \ltimes \mathbf{U}$*, where* $\mathbf{T}$ *is a torus and* $\mathbf{U}$ *is unipotent. Let* $A \in \mathfrak{g}_F$ *be a formal connection. Write* $A = A_{\mathbf{T}} + A_{\mathbf{U}}$ *for some* $A_{\mathbf{T}} \in$ *Lie*$(\mathbf{T})_F$ *and* $A_{\mathbf{U}} \in$ *Lie*$(\mathbf{U})_F$. *Let* b *be a positive integer such that* $b\,\pi\left((A_{\mathbf{T}})_{-1}\right) \in X_*(\mathbf{T})$. *Then there exists* $x \in \mathbf{G}(F_b)$ *such that* $x \cdot A = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1} C$ *with:*

(1) $r_j \in \mathbb{Z}_{<-1}$ *for all* j.

(2) $D_j \in Lie(\mathbf{T})$ *for all* j.

(3) $[D_j, C] = 0$ *for all* j.

(4) $\pi(C_s) = 0$.

Proof. The structure of the proof is very similar to the argument in Proposition 3.26. Write $A_{\mathbf{T}} = \sum_{j=-q}^{\infty} t^j D_j$. By Proposition 4.4 (a), we can find $g \in \mathbf{T}(F)$ with $g \cdot A_{\mathbf{T}} = \sum_{j=-q}^{-1} t^j D_j$. Set $\mu := b\,\pi\,(D_{-1}) \in X_*(\mathbf{T})$. Then we have $(t^{\frac{1}{b}\mu}\, g) \cdot A_{\mathbf{T}} = \sum_{j=-q}^{-2} D_j\, t^j \,+\, t^{-1}\, C_{\mathbf{T}}$ for some $C_{\mathbf{T}} \in \mathrm{Lie}(\mathbf{T})$ with $\pi(C_{\mathbf{T}}) = 0$.

Replace A with $B := (t^{\frac{1}{b}\mu}\, g) \cdot A$. We know that $B = \sum_{j=-q}^{-2} D_j\, t^j + t^{-1}\, C_{\mathbf{T}} + B_{\mathbf{U}}$ for some $B_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})_{F_b}$. By lifting to the b-ramified cover, we can assume that $B_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})_F$. We claim that we can find $u \in \mathbf{U}(F)$ such that $u \cdot B = \sum_{j=-q}^{-2} D_j\, t^j \,+\, t^{-1}\, C_{\mathbf{T}} + t^{-1}\, C_{\mathbf{U}}$ with $C_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})$ and $[C_{\mathbf{T}}, C_{\mathbf{U}}] = [D_j, C_{\mathbf{U}}] = 0$ for all j. We will show this by induction on the dimension of $\mathbf{U}$.

The base case is $\mathbf{U} = \mathbb{G}_a$. Then, $\mathbf{T}$ acts on $\mathbf{U}$ by a character $\chi : \mathbf{T} \longrightarrow \mathbb{G}_m$. For $u = \sum_{j=r}^{\infty} u_j\, t^j \in \mathbf{U}(F)$, we have

$$u \cdot B = t^{-1}\, C_{\mathbf{T}} \,+\, B_{\mathbf{U}} \,-\, \sum_{j=r-q}^{\infty} \left[(d\chi(C_{\mathbf{T}}) - j)\, u_j + \sum_{i=2}^{q} d\chi(D_i) u_{j+i-1} \right] t^{j-1}$$

We have two cases

(1) Suppose that $d\chi(D_j) \neq 0$ for some j. Then, we can solve the recurrence

$$(d\chi(C_{\mathbf{T}}) - j)\, u_j + \sum_{i=2}^{q} d\chi(D_i) u_{j+i} = B_{j-1}$$

with initial values $u_j = 0$ for $j \ll 0$. This yields an element $u \in \mathbf{U}(F)$ with $u \cdot B = \sum_{j=-q}^{-2} D_j\, t^j \,+\, t^{-1}\, C_{\mathbf{T}}$.

(2) Suppose that $d\chi(D_j) = 0$ for all j. The argument for the base case in Proposition 3.26 shows that there is an element $u \in \mathbf{U}(F)$ such that $u \cdot B = \sum_{j=-q}^{-2} D_j\, t^j + t^{-1}\, C_{\mathbf{T}} + t^{-1}\, C_{\mathbf{U}}$ for some $C_{\mathbf{U}} \in \mathrm{Lie}(\mathbf{U})$ satisfying $[C_{\mathbf{T}}, C_{\mathbf{U}}] = 0$. Notice that we have $[D_j, C_{\mathbf{U}}] = d\chi(D_j)\, C_{\mathbf{U}} = 0$ by assumption. So we are done in this case.

The base case follows.

Let's proceed with the induction step. We can decompose the action of the split torus $\mathbf{T}$ on the vector space $Z_{\mathbf{U}}$ into one-dimensional spaces. Let $\mathbf{H} \cong \mathbb{G}_a \leq Z_{\mathbf{U}}$ be one of these eigenspaces. Let s be a $\mathbf{T}$-equivariant section of the morphism of schemes $\mathbf{U} \longrightarrow \mathbf{U}/\,\mathbf{H}$ as in the proof of Proposition 3.26.

Let $\overline{B}$ be the image of B in the quotient $\text{Lie}(\mathbf{U}/\,\mathbf{H})_F$. By the induction hypothesis, we can find $\overline{u} \in \mathbf{U}/\,\mathbf{H}(F)$ such that $\overline{u} \cdot \overline{B} = \sum_{j=-q}^{-2} D_j\, t^j + t^{-1}\, C_{\mathbf{T}} + t^{-1}\, \overline{E}$ for some $\overline{E} \in \text{Lie}\,(\mathbf{U}/\,\mathbf{H})$ with $[D_j, \overline{E}] = [C_{\mathbf{T}}, \overline{E}] = 0$. We can then write

$$s(\overline{u}) \cdot B = \sum_{j=-q}^{-2} D_j\, t^j \; + \; t^{-1}\, C_{\mathbf{T}} + t^{-1}\, ds(\overline{E}) + B_{\mathbf{H}}$$

for some $B_{\mathbf{H}} \in \text{Lie}(\mathbf{H})_F$. Since s is $\mathbf{T}$-equivariant, we have $[ds(\overline{E}), D_j] = [ds(\overline{E}),\, C_{\mathbf{T}}] = 0$. We can use the base case for $\mathbf{H}$ in order to conclude. □

We end with a generalization of Proposition 3.29. We will use the same notation as in the regular case. Let $A = A^{\mathbf{T}} + A^{\mathbf{U}}$ be a formal connection with $A^{\mathbf{T}} \in \text{Lie}(\mathbf{T})_F$ and $A^{\mathbf{U}} \in \text{Lie}(\mathbf{U})_F$. Write $A^{\mathbf{T}} = \sum_{j=-q}^{-1} A_j^{\mathbf{T}}\, t^j \; + \; \sum_{j=p}^{\infty} A_j^{\mathbf{T}}\, t^j$ for some $q, p \geq 0$. Also, write $A^{\mathbf{U}} = \sum_{j=m}^{\infty} A_j^{\mathbf{U}}\, t^j$.

Proposition 5.2. *Keep the same notation as above. Assume that* $\mathbf{U}$ *has nilpotency class* n.

(i) Suppose that $m > L - 1$. *Then there exists* $x \in \mathbf{G}(\mathcal{O})$ *such that* $x \cdot A = \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j$. *More precisely, there exist* $x_{\mathbf{T}} \in \mathbf{T}(\mathcal{O})$ *with* $x_{\mathbf{T}} \equiv 1_{\mathbf{T}}$ *(mod* t^{p+1}*) and* $x_{\mathbf{U}} \in \mathbf{U}(\mathcal{O})$ *with* $x_{\mathbf{U}} \equiv 1_{\mathbf{U}}$ *(mod* t^{m+1}*) such that* $(x_{\mathbf{U}} x_{\mathbf{T}}) \cdot A = \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j$.

(ii) Suppose that $m \leq L - 1$. *Then the* $\mathbf{G}(F)$*-gauge equivalence class of* A *is determined by the coefficients* $A_j^{\mathbf{T}}$ *for* $-q \leq j < (n+1)(|m| - 1) + L$ *and* $A_j^{\mathbf{U}}$ *for* $-q \leq j < n(|m| - 1) + L$. *More precisely, suppose that there is another connection* B *and an integer* $k \geq n(|m| - 1) + L$ *satisfying* $A^{\mathbf{T}} \equiv B^{\mathbf{T}}$ $\left(\textit{mod}\; t^{k+|m|-1}\right)$ *and* $A^{\mathbf{U}} \equiv B^{\mathbf{U}}$ $\left(\textit{mod}\; t^{k}\right)$. *Then, there exists* $x \in \mathbf{G}(\mathcal{O})$ *with* $x \equiv 1$ $\left(\textit{mod}\; t^{k-n|m|+n+1}\right)$ *such that* $x \cdot A = B$.

Proof. The proof is similar in spirit to the argument for Proposition 3.29, but it involves an extra subtle twist to deal with the negative powers.

(i) Just as in Proposition 3.29, we can find $x_{\mathbf{T}} \in \mathbf{T}(\mathcal{O})$ with $x_{\mathbf{T}} \equiv 1_{\mathbf{T}}$ $(\textit{mod}\; t^{p+1})$ such that

$$C := x_{\mathbf{T}} \cdot A = \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j \; + \; C^{\mathbf{U}}$$

for some $C^{\mathbf{U}} \in \mathfrak{u}_{\mathcal{O}}$. Moreover we have $C^{\mathbf{U}} \equiv 0$ $(\textit{mod}\; t^m)$. We claim that there exists $u \in \mathbf{U}(\mathcal{O})$ with $u \equiv 1_{\mathbf{U}}$ $(\textit{mod}\; t^{m+1})$ such that $u \cdot C = \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j$. This claim finishes the proof of part (i).

In order to prove the claim, we will actually show something stronger. Let us fix some notation. By [BS68] Corollary 9.12, there is a $\mathbf{T}$-equivariant map of $\mathbf{k}$-schemes $\psi_{\mathbf{U}} : \mathbf{U} \longrightarrow \mathfrak{u}$. We can define this map so that the following diagram commutes

$$\begin{tikzcd} \mathbf{U} \arrow[r] \arrow[d, "\psi_{\mathbf{U}}"] & \mathbf{U}/\,Z_{\mathbf{U}} \arrow[d, "\psi_{\mathbf{U}/\,Z_{\mathbf{U}}}"] \\ \mathfrak{u} \arrow[r] & \mathfrak{u}/\,\mathfrak{z} \end{tikzcd}$$

Here $Z_{\mathbf{U}}$ is the center of $\mathbf{U}$ and $\mathfrak{z} = \mathrm{Lie}(Z_{\mathbf{U}})$. Notice that $Z_{\mathbf{U}}$ is just a direct sum of copies of $\mathbb{G}_a$. The corresponding map $\psi_{Z_{\mathbf{U}}}$ can be taken to be the usual identification of a vector space with its tangent space at the identity. By iterating, we can arrange so that we get a corresponding compatibility at each step of the upper central series of $\mathbf{U}$.

Recall that we have a weight decomposition $\mathfrak{u} = \bigoplus_{i=1}^{l} \mathfrak{u}_{\chi_i}$. Via the isomorphism $\psi_{\mathbf{U}}$, we can get a decomposition $\mathbf{U} = \prod_{\chi_i} \mathbf{U}_{\chi_i}$ as a product of schemes. For $u \in \mathbf{U}(\mathbf{k})$, we will denote by u_{χ_i} the corresponding component in $\mathbf{U}_{\chi_i}$.

For each i, define a_i to be the biggest positive integer j such that $d\chi_i\left(A^{\mathbf{T}}_{-j}\right) \neq 0$. If $d\chi_i\left(A^{\mathbf{T}}_{-j}\right) = 0$ for all $j > 0$, we set $a_i = 1$. Then, we claim that we can find $u \in \mathbf{U}(\mathcal{O})$ with $u_{\chi_i} \equiv 1_{\mathbf{U}} \; (mod\; t^{m+a_i})$ such that $u \cdot C = \sum_{-q}^{-1} A^{\mathbf{T}}_j t^j$. We will prove this stronger claim by induction on the nilpotency class of $\mathbf{U}$.

For the base case $n = 0$, we have $\mathbf{U} \cong \mathbb{G}_a^d$ for some d. By decomposing into one-dimensional $\mathbf{T}$-modules and looking at each coordinate, we can reduce to the case $d = 1$. So we have a single weight space $\mathfrak{u}_{\chi_i}$. This case amounts to solving a recurrence as in the computation for the base case in Proposition 5.1. We want to find $u = \sum_{j=0}^{\infty} u_j t^j$ satisfying

$$\left(d\chi_i(A^{\mathbf{T}}_{-1}) - j\right) u_j + \sum_{k=2}^{q} d\chi_i(A^{\mathbf{T}}_{-k}) u_{j+k} = C^{\mathbf{U}}_{j-1}$$

By the definition of a_i, this is the same as

$$\left(d\chi_i(A^{\mathbf{T}}_{-1}) - j\right) u_j + \sum_{k=2}^{a_i} d\chi_i(A^{\mathbf{T}}_{-k}) u_{j+k} = C^{\mathbf{U}}_{j-1}$$

There are two different cases.

(1) If $a_i = 1$, then the recurrence reduces to

$$\left(d\chi_i(A^{\mathbf{T}}_{-1}) - j\right) u_j = C^{\mathbf{U}}_{j-1}$$

The claim follows by the argument for the base case in Proposition 3.29.

(2) Suppose that $a_i \neq 1$. We know that $d\chi_i(A^{\mathbf{T}}_{-a_i}) \neq 0$. We can solve the recurrence by rewriting

$$d\chi_i(A^{\mathbf{T}}_{-a_i})\, u_{j+a_i} = C^{\mathbf{U}}_{j-1} - \left(d\chi_i(A^{\mathbf{T}}_{-1}) - j\right) u_j - \sum_{k=2}^{a_i-1} d\chi_i(A^{\mathbf{T}}_{-k}) u_{j+k}$$

Since $C^{\mathbf{U}}_j = 0$ for all $j \leq m-1$, we can set $u_j = 0$ for all $j \leq m + a_i$. Then we can solve for the rest of the u_j using the recursion formula above.

Let's proceed with the induction step. Notice that $\mathfrak{z}$ is a direct sum of some one-dimensional $\mathbf{T}$-submodules of $\mathfrak{u}$. We can get an identification of

$\mathfrak{u}/\,\mathfrak{z}$ with the direct sum of some choice of remaining one-dimensional $\mathbf{T}$-submodules. This way we get a $\mathbf{T}$-equivariant inclusion $\mathfrak{u}/\,\mathfrak{z} \hookrightarrow \mathfrak{u}$. We can get a $\mathbf{T}$-equivariant section $s : \mathbf{U}/\,Z_{\mathbf{U}} \longrightarrow \mathbf{U}$ defined by the composition

$$s : \mathbf{U}/\,Z_{\mathbf{U}} \xrightarrow{\psi_{\mathbf{U}/\,Z_{\mathbf{U}}}} \mathfrak{u}/\,\mathfrak{z} \hookrightarrow \mathfrak{u} \xrightarrow{\psi_{\mathbf{U}}^{-1}} \mathbf{U}$$

Let $\overline{C}$ be the image of C in the quotient $\mathrm{Lie}(\mathbf{T} \ltimes \mathbf{U}/\,Z_{\mathbf{U}})_{F_b}$. By the induction hypothesis, there exists $\overline{x} \in \mathbf{U}/\,Z_{\mathbf{U}}(\mathcal{O})$ such that $\overline{x}_{\chi_i} \equiv 1 \pmod{t^{m+a_i}}$ and $\overline{x} \cdot \overline{C} = \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j$. By the $\mathbf{T}$-equivariance of s, we must then have $s(\overline{x}) \cdot C = \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j \,+\, D_{Z_{\mathbf{U}}}$ for some $D_{Z_{\mathbf{U}}} \in \mathrm{Lie}(Z_{\mathbf{U}})_F$. By definition

$$s(\overline{x}) \cdot C \;=\; \sum_{-q}^{-1} t^j \mathrm{Ad}(s(\overline{x}))A_j^{\mathbf{T}} \,+\, \mathrm{Ad}(s(\overline{x}))C^{\mathbf{U}} \,+\, ds(\overline{x})s(\overline{x})^{-1}$$

Since $s(\overline{x}) \equiv s(\overline{x})^{-1} \equiv 1 \pmod{t^{m+1}}$, it follows that $ds(\overline{x})s(\overline{x})^{-1} \equiv 0 \pmod{t^{m+1}}$. Also $\mathrm{Ad}(s(\overline{x}))C^{\mathbf{U}} \equiv C^{\mathbf{U}} \pmod{t^{m+1}}$, because by assumption $C_{\mathbf{U}} \in \mathfrak{u}_{\mathcal{O}}$. We are left to study $\mathrm{Ad}(s(\overline{x}))A_j^{\mathbf{T}}$.

Consider the map of $\mathbf{k}$-schemes $\varphi_j : \mathbf{U} \longrightarrow \mathfrak{u}$ given by $\varphi_j(u) := \mathrm{Ad}(u)A_j^{\mathbf{T}} - A_j^{\mathbf{T}}$. By construction φ_j is $\mathbf{T}$-equivariant. This means that it must respect the decomposition into weight spaces. In other words, the χ_i-coordinate of $\varphi_j(u)$ is given by $\varphi_j(u_{\chi_i})$. In particular, this means that

$$\mathrm{Ad}(s(\overline{x}))A_j^{\mathbf{T}} = A_j^{\mathbf{T}} \,+\, \sum_{i=1}^{l} \left(\mathrm{Ad}(s(\overline{x})_{\chi_i})A_j^{\mathbf{T}} - A_j^{\mathbf{T}} \right)$$

We have that $\mathrm{Ad}(s(\overline{x})_{\chi_i})A_j^{\mathbf{T}} = A_j^{\mathbf{T}}$ whenever $d\chi_i \left(A_j^{\mathbf{T}}\right) = 0$. By definition this happens whenever $-j > a_i$. So we get

$$\mathrm{Ad}(s(\overline{x}))A_j^{\mathbf{T}} = A_j^{\mathbf{T}} \,+\, \sum_{-j \leq a_i} \left(\mathrm{Ad}(s(\overline{x})_{\chi_i})A_j^{\mathbf{T}} - A_j^{\mathbf{T}} \right)$$

Suppose that $-j \leq a_i$. By assumption $s(\overline{x})_{\chi_i} \equiv 1 \pmod{t^{m+a_i}}$, so in particular $s(\overline{x})_{\chi_i} \equiv 1 \pmod{t^{m-j}}$. Hence we have $\mathrm{Ad}(s(\overline{x})_{\chi_i})A_j^{\mathbf{T}} \equiv A_j^{\mathbf{T}} \pmod{t^{m-j}}$. The sum above becomes

$$\mathrm{Ad}(s(\overline{x}))A_j^{\mathbf{T}} \;\equiv\; A_j^{\mathbf{T}} \pmod{t^{m-j}}$$

Hence $t^j\, \mathrm{Ad}(s(\overline{x})A_j^{\mathbf{T}} \;\equiv\; t^j\, A_j^{\mathbf{T}} \pmod{t^m}$. We can put together all of the discussion above to conclude that

$$s(\overline{x}) \cdot C \;\equiv\; \sum_{-q}^{-1} A_j^{\mathbf{T}}\, t^j \,+ C^{\mathbf{U}} \;=\; C \pmod{t^m}$$

Therefore $D_{Z_{\mathbf{U}}} \equiv 0 \pmod{t^m}$. Now we can conclude by using the base case for $Z_{\mathbf{U}}$.

(ii) The hypothesis implies that we have equality of singular parts $\sum_{j=-q}^{-1} B_j^{\mathbf{T}}\, t^j = \sum_{j=-q}^{-1} A_j^{\mathbf{T}}\, t^j$. The proof of Proposition 3.12 shows that there exist $x_{\mathbf{T}} \in \mathbf{T}(\mathcal{O})$ with $x_{\mathbf{T}} \equiv 1_{\mathbf{T}}\ (mod\ t^{p+1})$ such that $x_{\mathbf{T}} \cdot A^{\mathbf{T}} = B^{\mathbf{T}}$. Set $C := x_{\mathbf{T}} \cdot A$. We have $C = B^{\mathbf{T}} + \mathrm{Ad}(x_{\mathbf{T}})A^{\mathbf{U}}$. Define $C^{\mathbf{U}} := \mathrm{Ad}(x_{\mathbf{T}})A^{\mathbf{U}}$. We know that $C^{\mathbf{U}} \equiv A^{\mathbf{U}}\ \left(mod\ t^k\right)$, because $x_{\mathbf{T}} \equiv 1\ \left(mod\ t^{k+|m|}\right)$ and $A^{\mathbf{U}} \in t^m \mathfrak{u}_{\mathcal{O}}$. Therefore $C^{\mathbf{U}} \equiv B^{\mathbf{U}}\ \left(mod\ t^k\right)$ by assumption.

Let s, $\mathbf{U}_{\chi_i}$ and a_i be defined as in part (i). We claim that there exists $u \in \mathbf{U}(\mathcal{O})$ with $u_{\chi_i} \equiv 1\ \left(mod\ t^{k-n|m|+n+a_i}\right)$ such that $u \cdot C = B$. This implies that $u \equiv 1\ \left(mod\ t^{k-n|m|+n+1}\right)$, so this claim concludes the proof of part (ii). In order to prove the claim, we will induct on the nilpotency class of $\mathbf{U}$. The base case $n = 0$ follows again from the explicit computation done in Proposition 5.1, we ommit the details.

Let's proceed with the induction step. Let $\overline{C}$ and $\overline{B}$ denote the images of C and B in the quotient $\mathrm{Lie}(\mathbf{T} \ltimes \mathbf{U}/\, Z_{\mathbf{U}})_F$. By the induction hypothesis, there exists $\overline{x} \in \mathbf{U}/\, Z_{\mathbf{U}}(\mathcal{O})$ with $\overline{x}_{\chi_i} \equiv 1\ \left(mod\ t^{k-(n-1)|m|+n-1+a_i}\right)$ such that $\overline{x} \cdot \overline{C} = \overline{B}$. We can now write $s(\overline{x}) \cdot C = ds\left(\overline{B}\right) + E_{Z_{\mathbf{U}}}$ and $B = ds\left(\overline{B}\right) + K_{Z_{\mathbf{U}}}$ for some $E_{Z_{\mathbf{U}}}, F_{Z_{\mathbf{U}}} \in \mathrm{Lie}(Z_{\mathbf{U}})_F$. By definition

$$s(\overline{x}) \cdot C \;=\; \sum_{j=-q}^{\infty} t^j \mathrm{Ad}(s(\overline{x}))B_j^{\mathbf{T}} \;+\; \mathrm{Ad}(s(\overline{x}))C^{\mathbf{U}} \;+\; ds(\overline{x})s(\overline{x})^{-1}$$

Since $s(\overline{x}) \equiv 1\ \left(mod\ t^{k-(n-1)|m|+n}\right)$, it follows that
$t^j\, \mathrm{Ad}(s(\overline{x}))B_j^{\mathbf{T}} \equiv t^j\, B_j^{\mathbf{T}}\ \left(mod\ t^{k-(n-1)|m|+n}\right)$ for all $j \geq 0$. The same reasoning as in part (i) shows that $t^j\, \mathrm{Ad}(s(\overline{x}))B_j^{\mathbf{T}} \equiv t^j\, B_j^{\mathbf{T}}\ \left(mod\ t^{k-(n-1)|m|+n-1}\right)$ for all $j < 0$. Also we know that $\mathrm{Ad}(s(\overline{x})C^{\mathbf{U}} \equiv C^{\mathbf{U}}\ \left(mod\ t^{k-n|m|+n}\right)$, because $s(\overline{x}) \equiv 1\ \left(mod\ t^{k-(n-1)|m|+n}\right)$ and $C_{\mathbf{U}} \in t^m \mathfrak{u}_{\mathcal{O}}$. We conclude that

$$ds\left(\overline{B}\right) + E_{Z_{\mathbf{U}}} \;=\; s(\overline{x}) \cdot C \;\equiv\; B^{\mathbf{T}} \,+\, C^{\mathbf{U}} \;=\; C\ \left(mod\ t^{k-n|m|+n}\right)$$

Since $k \geq k - n|m|$, we have $C \equiv B\ \left(mod\ t^{k-n|m|}\right)$. It follows that $E_{Z_{\mathbf{U}}} \equiv K_{Z_{\mathbf{U}}}\ \left(mod\ t^{k-n|m|+n}\right)$. Now by the base case we can find $y \in Z_{\mathbf{U}}(\mathcal{O})$ with $y_{\chi_i} \equiv 1\ \left(mod\ t^{k-n|m|+n+a_i}\right)$ such that $(y\, s(\overline{x})) \cdot C = B$. By the definition of $\mathbf{U}_{\chi_i}$ and its compatibility with the center, we can see that $(y\, s(\overline{x}))_{\chi_i} \equiv 1\ \left(mod\ t^{k-n|m|+n+a_i}\right)$. The claim follows.

□

5.2 Irregular connections for arbitrary linear algebraic groups

Theorem 5.3. *Let $\mathbf{G}$ be a connected linear algebraic group. Fix a Levi subgroup $\mathbf{L}$ and a maximal torus $\mathbf{T} \subset \mathbf{L}$. Let $A \in \mathfrak{g}_{\overline{F}}$ be a formal connection. Then there exists $x \in \mathbf{G}(\overline{F})$ such that $x \cdot A = \sum_{j=1}^{l} D_j\, t^{r_j} \,+\, t^{-1}\, C$ with*

(1) $r_j \in \mathbb{Q}_{<-1}$ for all j.

(2) $D_j \in Lie(\mathbf{T})$ for all j.

(3) $[D_j, C] = 0$ *for all* j.

(4) $C_s \in \mathfrak{D}$.

(5) $[C_s, C] = 0$.

Proof. The same steps as in the proof of Theorem 3.32 reduce the result to the solvable case (Proposition 5.1). $\square$

A connection of the form $B = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}\, C$ satisfying conditions (1)-(3) above is said to be in canonical form. Let us formulate some uniqueness results for such irregular canonical forms. Before doing this, we need a lemma.

Lemma 5.4. *Let* $B = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}\, C$ *and* $B' = \sum_{j=1}^{s} D'_j\, t^{r'_j} + t^{-1}\, C'$ *be two connections in canonical form. Suppose that* $x \in \mathbf{G}(\overline{F})$ *satisfies* $x \cdot B = B'$. *Then all the following statements are true*

(1) $l = s$ *and* $r_j = r'_j$.

(2) $Ad(x) D_j = D'_j$ *for all* j.

(3) $x \cdot (t^{-1}\, C) = t^{-1}\, C'$.

Proof. If we know both (1) and (2), then part (3) follows. So we will focus on the first couple of statements. By lifting everything to a ramified cover, we can assume that $x \in \mathbf{G}(F)$. Choose a faithful representation $\mathbf{G} \hookrightarrow \mathrm{GL}_{\mathbf{n}}$. We can view $x \in \mathrm{GL}_{\mathbf{n}}(F)$ and $B, B' \in \mathfrak{gl}_n(\overline{F})$.

To simplify notation, let us add some trivial D_js and D'_js so that we have the same indexes and exponents for both B_{irr} and B'_{irr}. We therefore write $B = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}\, C$ and $B' = \sum_{j=1}^{l} D'_j\, t^{r_j} + t^{-1}\, C'$. Now the D_js and D'_js are (possibly 0) semisimple elements in $\mathfrak{g}$. We claim that $\mathrm{Ad}(x) D_j = D'_j$ for all j. Notice that this claim would imply that none of the new D_j and D'_j are 0. This would mean that we didn't actually add any extra terms. So both (1) and (2) would follow. We are left to show the claim.

Let us consider the linear transformation W in $\mathrm{End}(\mathfrak{gl}_n)\,(\overline{F})$ given by $W\, v = B'v \,-\, vB$ for all $v \in \mathfrak{gl}_n$. We can write $W = \sum_{j=1}^{l} W_j\, t^{r_j} \,+\, t^{-1}\, U$, where

$$W_j \in \mathrm{End}(\mathfrak{gl}_n) \text{ is given by } W_j\, v := D'_j v - v D_j$$
$$U \in \mathrm{End}(\mathfrak{gl}_n) \text{ is given by } U\, v := C'v - vC$$

The W_js are semisimple by definition. Also we have that the W_js and U pairwise commute. Therefore there is a simultaneous spectral decomposition $\mathfrak{gl}_n = \bigoplus_{\vec{\lambda}} (\mathfrak{gl}_n)_{\vec{\lambda}}$ for the W_js, where $\vec{\lambda} = (\lambda_j)_{j=1}^{l}$ ranges over a set of l-tuples of eigenvalues of the W_js. Note that W preserves this spectral decomposition, because U commutes with all W_js.

The condition $x \cdot B = B'$ can be expressed as $\frac{d}{dt} x = W\, x$. Here we are viewing x as an invertible matrix in $\mathfrak{gl}_n(F)$. We can restrict to the $\vec{\lambda}$-eigenspace and use the decomposition for W in order to see that the component $x_{\vec{\lambda}} \in (\mathfrak{gl}_n)_{\vec{\lambda}}$ of x satisfies

$$\frac{d}{dt} x_{\vec{\lambda}} = \sum_{j=1}^{l} \lambda_j \, t^{r_j} \, x_{\vec{\lambda}} \, + \, t^{-1} \, U \, x_{\vec{\lambda}}$$

Recall that $r_j < -1$ for all j. By comparing the smallest exponent of t in both sides, we conclude that $x_{\vec{\lambda}} = 0$ unless $\vec{\lambda} = \vec{0}$. Hence $x \in (\mathfrak{gl}_n)_{\vec{0}}(F)$. This means that $\mathrm{Ad}(x) D_j = D'_j$ for all j. □

As a consequence, we get the following uniqueness result for all irregular canonical forms that satisfy (1)-(5) as in Therorem 5.3.

Theorem 5.5. *Let $\mathbf{G}$ be a connected linear algebraic group. Fix a Levi subgroup $\mathbf{L}$ and a maximal torus $\mathbf{T} \subset \mathbf{L}$. Let $A = \sum_{j=1}^{l} D_j \, t^{r_j} \, + \, t^{-1} \, C$ and $B = \sum_{j=1}^{l} D'_j \, t^{r'_j} \, + \, t^{-1} \, C'$ be two connections in canonical form. Suppose that C_s, $C'_s \in \mathfrak{D}$ and $[C_s, C] = [C'_s, C'] = 0$. If there exists $x \in \mathbf{G}(\overline{F})$ with $x \cdot A = B$, then we have*

(1) $C_s = C'_s$.

(2) $x \in Z_{\mathbf{G}}(C_s)(\mathbf{k})$.

(3) $Ad(x) \, D_j = D'_j$ for all j.

Proof. This follows from Lemma 5.4 combined with Proposition 3.34. □

We conclude this section with a determinacy result for arbitrary linear algebraic groups.

Proposition 5.6. *Let $\mathbf{G}$ be connected linear algebraic group. Let $A \in \mathfrak{g}_F$ be a connection. There exist a positive integer n such that all connections $C \in \mathfrak{g}_F$ satisfying $C \equiv A \pmod{t^n}$ are $\mathbf{G}(\overline{F})$-gauge equivalent to A.*

Proof. This follows from the corresponding determinacy results for reductive groups (Proposition 4.23) and solvable groups (Proposition 5.2) via a reduction as in the proof of Theorem 5.3. □

5.3 Galois cohomology for irregular connections

For this section $\mathbf{G}$ will be a connected linear algebraic group. We will fix a choice of Levi subgroup $\mathbf{L} \subset \mathbf{G}$ and maximal torus $\mathbf{T} \subset \mathbf{L}$. Let $B = \sum_{j=1}^{l} D_j \, t^{r_j} + t^{-1} C$ be a connection in canonical form with $C_s \in \mathfrak{D}$ and $[C_s, C] = 0$, as in the statement of Theorem 5.5. If $B_{\mathrm{irr}} \neq 0$, then we don't necessarily have $B \in \mathfrak{g}_F$. Suppose that B is in $\mathfrak{g}_{F_b}$, with b a given positive integer. Then we have a Galois action of $\mu_b \cong \mathrm{Gal}(F_b / F)$ on B by the formula $\gamma \cdot B = \sum_{j=1}^{l} \gamma^{-b r_j} \, D_j \, t^{r_j} \, + \, t^{-1} \, C$. Because this action is not necessarily trivial, we have to consider twisted cocyles in order to classify connections over $\mathrm{Spec}\, F$ with canonical form B.

Definition 5.7. *Let b be a natural number. Let $B = \sum_{j=1}^{l} D_j \, t^{r_j} + t^{-1} C \, \in \mathfrak{g}_{F_b}$ be a connection in canonical form. A B-twisted μ_b-cocycle is a map $\phi : \mu_b \longrightarrow Z_G(C)$ satisfying*

1. *$Ad(\phi_\gamma)B = \gamma \cdot B$ for all $\gamma \in \mu_b$.*

2. *$\phi_{\gamma\gamma'} = \phi_\gamma \phi_{\gamma'}$ for all $\gamma, \gamma' \in \mu_b$.*

We can fix a compatible choice of generator ω_b of μ_b for all b positive, as we did in the regular case. Note that a B-twisted μ_b cocycle ϕ is completely determined by $\phi_{\omega_b} \in Z_G(C)$. This is a an element of finite order dividing b, and it satisfies $\mathrm{Ad}(\phi_{\omega_b})B = \omega_b \cdot B$. Conversely, for any element $\phi_{\omega_b} \in Z_G(C)$ satisfying $\mathrm{Ad}(\phi_{\omega_b})B = \omega_b \cdot B$ we can get a corresponding B-twisted cocycle.

Notice that the centralizer $Z_G(\{D_1, ..., D_l, C\})$ acts on the set of B-twisted μ_b-cocycles by conjugation. By the same type of general Galois cohomology argument as in the regular case, we get the following couple of propositions. The proofs are ommited.

Proposition 5.8 (Criterion for Descent to D^*)**.** *Let b be a natural number. Let $B = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}C \ \in \mathfrak{g}_{F_b}$ be a connection in canonical form with $C_s \in \mathfrak{D}$ and $[C_s, C] = 0$. Then B is equivalent to a connection in $\mathfrak{g}_F$ via an element of $\mathbf{G}(F_b)$ if and only there exists a B-twisted μ_b-cocycle.*

Proposition 5.9 (Classification of Connections over D^*)**.** *Let $B = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}C \ \in \mathfrak{g}_{F_b}$ be a connection in canonical form with $C_s \in \mathfrak{D}$ and $[C_s, C] = 0$. Suppose that B satisfies the equivalent statements in Proposition 5.8 above for some b. Then the set of equivalence classes of $\mathbf{G}$-connections over D^* that become gauge equivalent over $\operatorname{Spec} F_b$ are in bijection with the set of B-twisted μ_b-cocycles up to $Z_G(\{D_1, ..., D_l, C\})$-conjugacy.*

The correspondence in Proposition 5.9 can be described as follows. Let $\phi_{\omega_b} \in Z_{\mathbf{G}}(C)(\mathbf{k})$ such that $\mathrm{Ad}(\phi_{\omega_b})B = \omega_b \cdot B$. By the vanishing of $H^1_{\mathrm{Gal}(F)}(\mathbf{G})$, we can find an element $y \in \mathbf{G}(F_b)$ such that $\omega_b \cdot y = y\, \phi_{\omega_b}$. The connection associated to ϕ_{ω_b} will be $A = y \cdot B \ \in \mathfrak{g}_F$. Conversely, suppose that $A = y \cdot B$ is a connection in $\mathfrak{g}_F$ for some $y \in \mathbf{G}(F_b)$. We set $\phi_{\omega_b} := y^{-1}\ (\omega_b \cdot y)$.

As a consequence of this Galois cohomology classification, we can put a bound on the denominators of the levels r_j of a canonical form for a connection in $\mathfrak{g}_F$. Let W denote the Weyl group of $\mathbf{L}$ with respect to $\mathbf{T}$. A Coxeter element of W is an element of largest length in W. All Coxeter elements are conjugate to each other. The Coxeter number $h_{\mathbf{L}}$ of $\mathbf{L}$ is the order of a Coxeter element in W.

Proposition 5.10. *Let $A \in \mathfrak{g}_F$ be a formal connection. Let $B = \sum_{j=1}^{l} D_j\, t^{r_j} + t^{-1}C$ be a connection in canonical form that is $\mathbf{G}(\overline{F})$-gauge equivalent to A. Suppose that $C_s \in \mathcal{D}$ and $[C_s, C] = 0$. Let b be the smallest positive integer such that $B \in \mathfrak{g}_{F_b}$. Then*

(i) b divides a fundamental degree of Lie($\boldsymbol{L}$). In particular $b \leq h_{\mathbf{L}}$.

(ii) If $b = h_{\mathbf{L}}$, then $C_s \in$ Lie($Z^0_{\mathbf{L}}$).

Proof. Recall the notation $B_{irr} = \sum_{j=1}^{l} D_j\, t^{r_j}$. We have $\mathbf{G} = \mathbf{L} \ltimes \mathbf{U}$, where $\mathbf{U}$ is the unipotent radical. Write $\mathfrak{l} :=$ Lie($\mathbf{L}$) and $\mathfrak{u} := \mathbf{U}$. We can decompose $A = A_{\mathfrak{l}} + A_{\mathfrak{u}}$. It follows from the proof of Theorem 5.3 that B_{irr} is given by the irregular part of the canonical form of $A_{\mathfrak{l}}$. Therefore, we can assume without loss of generality that $\mathbf{G} = \mathbf{L}$.

By assumption, we have $B = \mathbf{G}(F_d) \cdot A$ for some $d|b$. By Proposition 5.8, we know that there exists a B-twisted μ_d-cocycle ϕ. This means in particular that $\mathrm{Ad}(\phi_{\omega_d})(B_{irr}+t^{-1}C_s) = \omega_d \cdot B_{irr}+t^{-1}C_s$. We can consider $B_{irr}+t^{-1}C_s$ as an element of $\mathrm{Lie}(\mathbf{T}_{\overline{F}})$. Also ϕ_{ω_d} can be viewed as an element of $\mathbf{G}(\overline{F})$. This means that $B_{irr} + t^{-1}$ and $\omega_d \cdot B_{irr} + t^{-1}$ are $\mathbf{G}(\overline{F})$-conjugate elements of $\mathrm{Lie}(\mathbf{T}_{\overline{F}})$. By [CM93] Chapter 2, there is an element $w \in W$ such that $w \cdot (B_{irr}t^{-1}C_s) = \omega_d \cdot B_{irr} + t^{-1}C_s$. By definition, b is the least positive integer such that $(\omega_d)^b \cdot B_{irr} = B_{irr}$. We conclude that some of the eigenvalues of w are primitive b roots of unity. It follows that b divides a fundamental degree of $\mathfrak{l}$ by [Spr74] Theorem 3.4. If $b = h_{\mathbf{L}}$, then w must be a Coxeter element by [Kan01] Theorem 32.2-C. Since $w \cdot C_s = C_s$, we must have $C_s \in \mathrm{Lie}(Z^0_{\mathbf{L}})$ by the lemma in page 76 of [Hum90]. $\square$

Remark 5.11. *This does not yield a bound on the ramification needed to put A into canonical form. For example, if A is regular then $B \in \mathfrak{g}_F$. But we have seen that it is sometimes necessary to pass to a ramified cover in order to put a a regular connection into canonical form.*

It should be remarked that part (i) of Proposition 5.10 was proven in [CK17] via the existence of oper structures for any connection [FZ10]. Here we note that there is a direct argument using some facts about Coxeter groups.

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